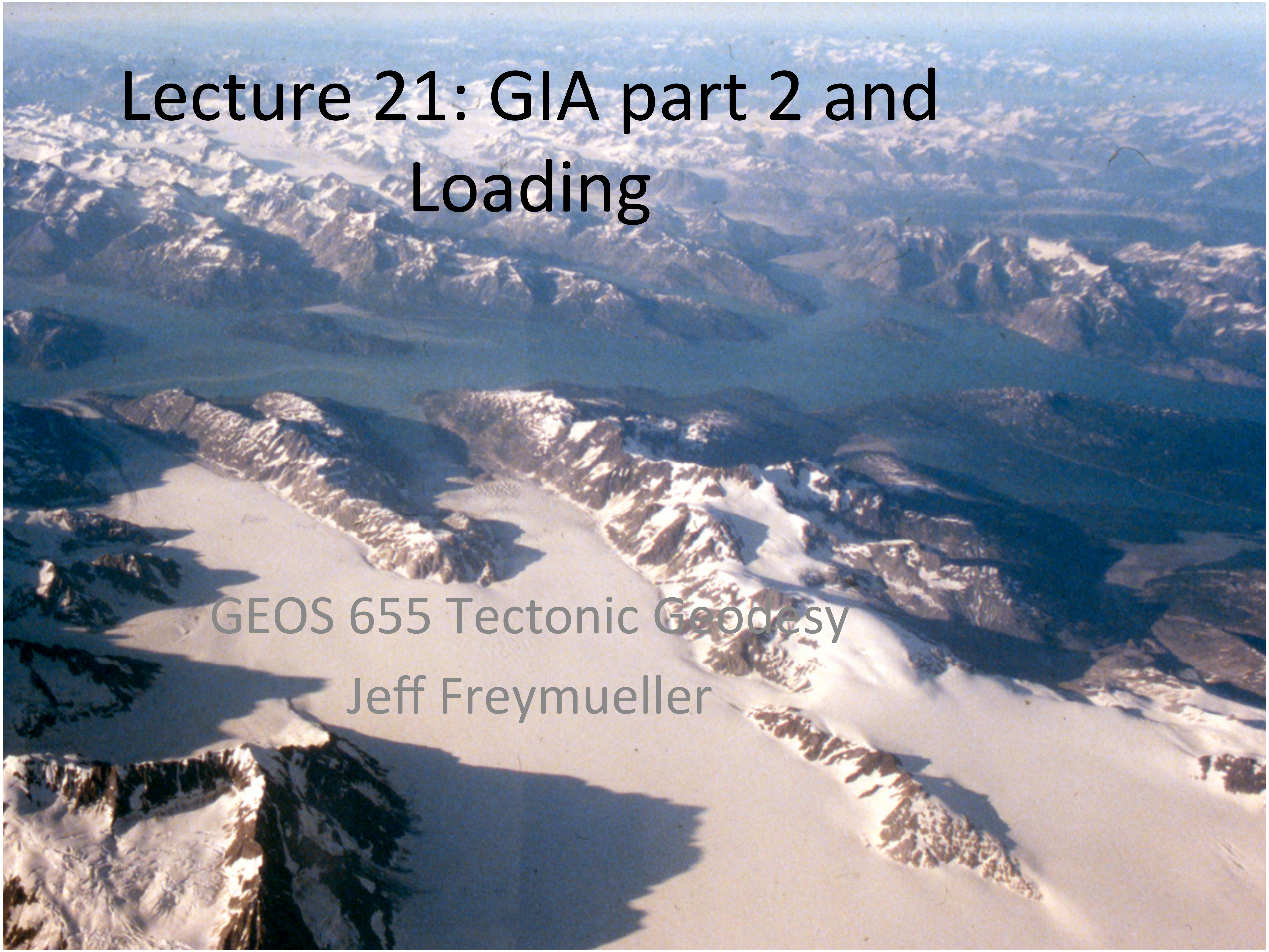
Lecture 21: GIA part 2 and Loading

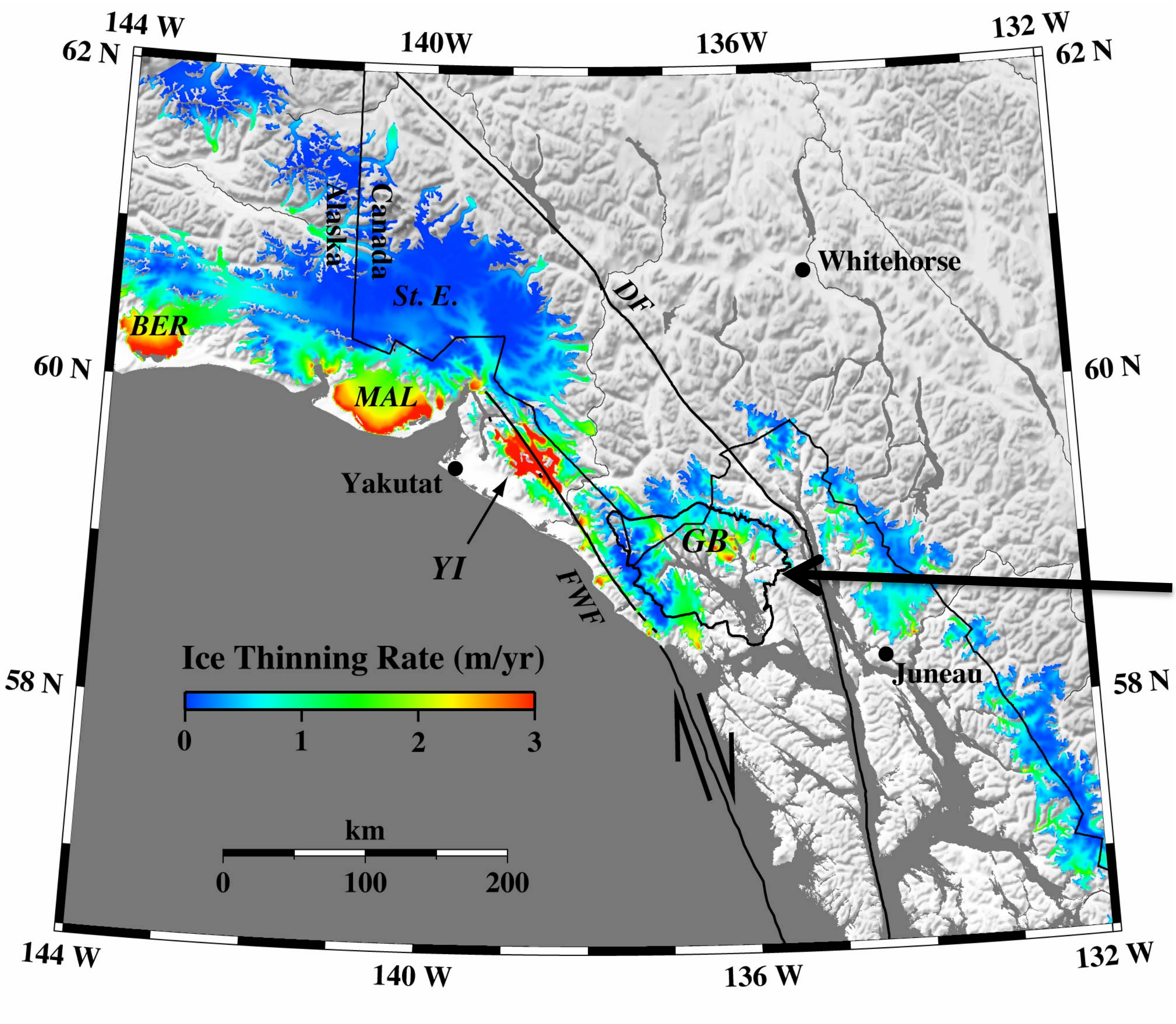
GEOS 655 Tectonic Geodesy

Jeff Freymueller



Average Regional Thinning Rate

1950s to 1990s, Arendt et al. (2002)



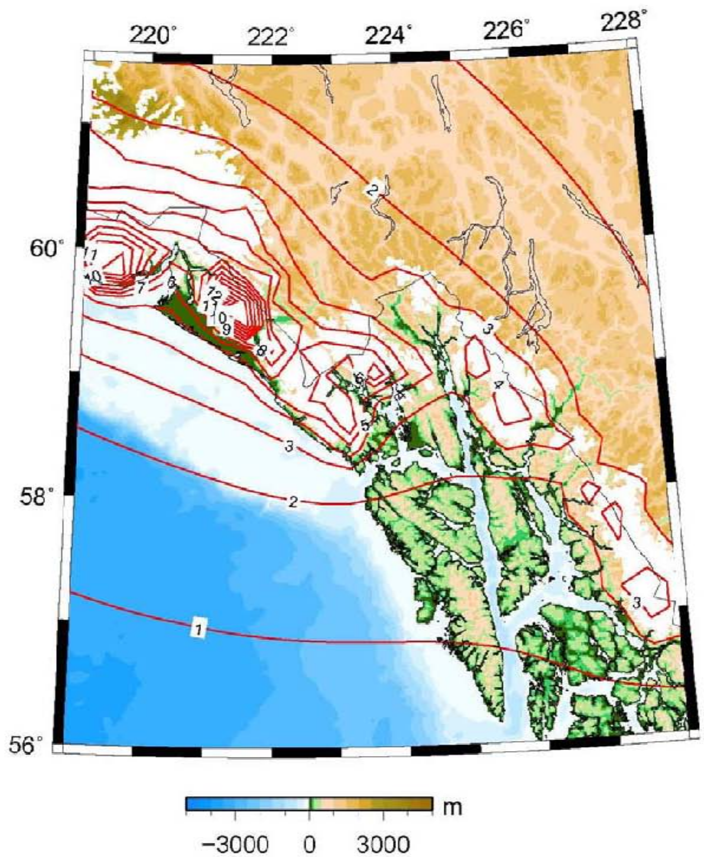
Regional: 5800 km^3 lost in 20th century. May be underestimated by a factor of 2 (Larsen et al., 2007)

Glacier Bay: 3000 km^3 lost in 19th century (Larsen et al., 2005)

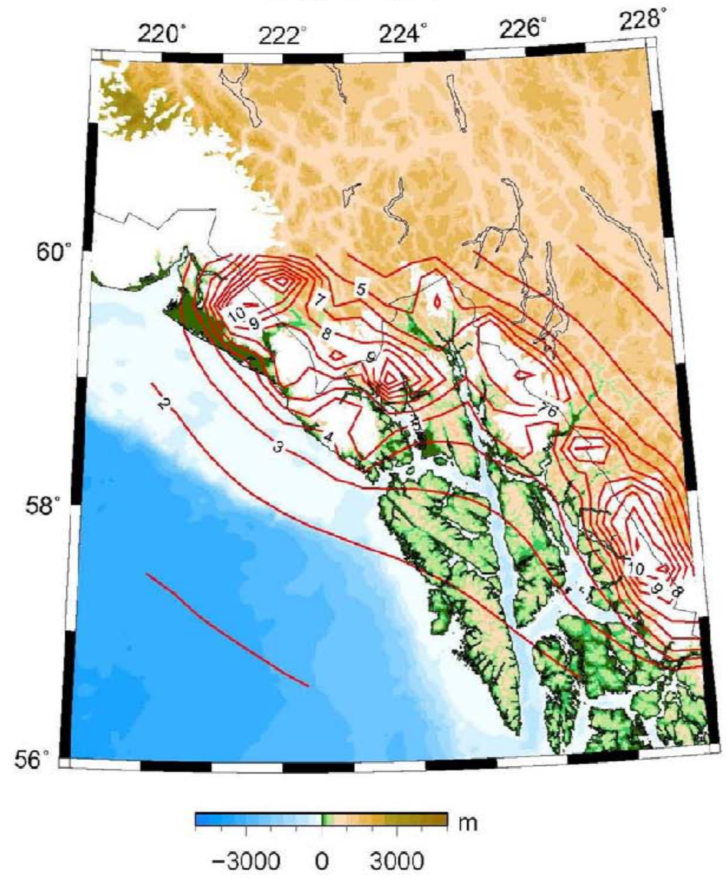
Elastic Response to Present-Day Ice Melting

Effect to the uplift rate

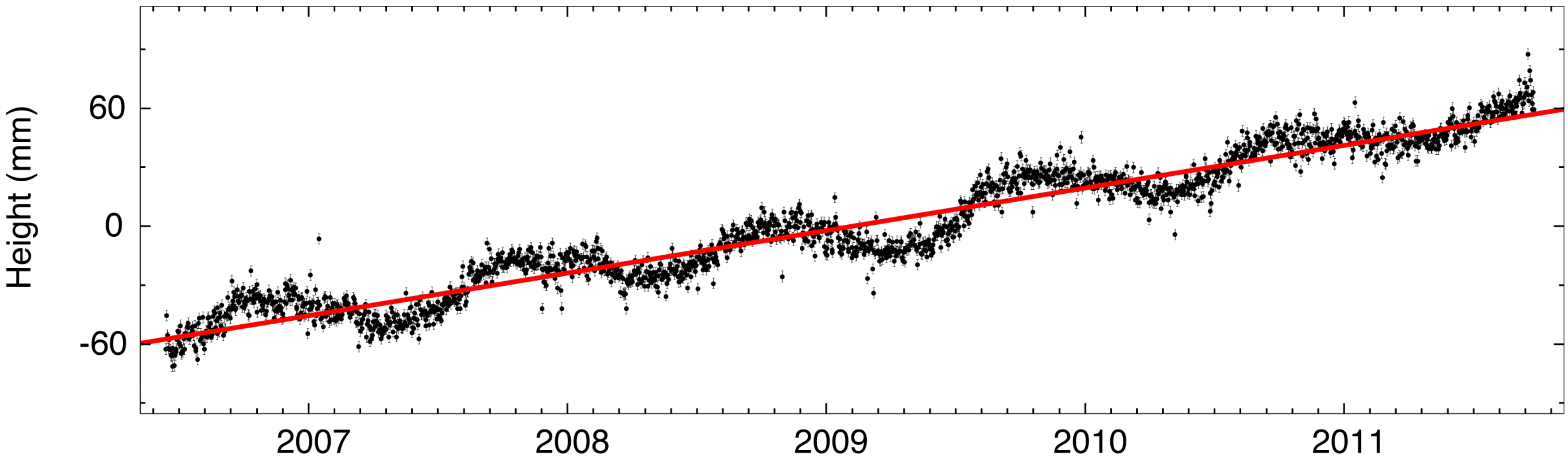
EPSL2005



JGR2007

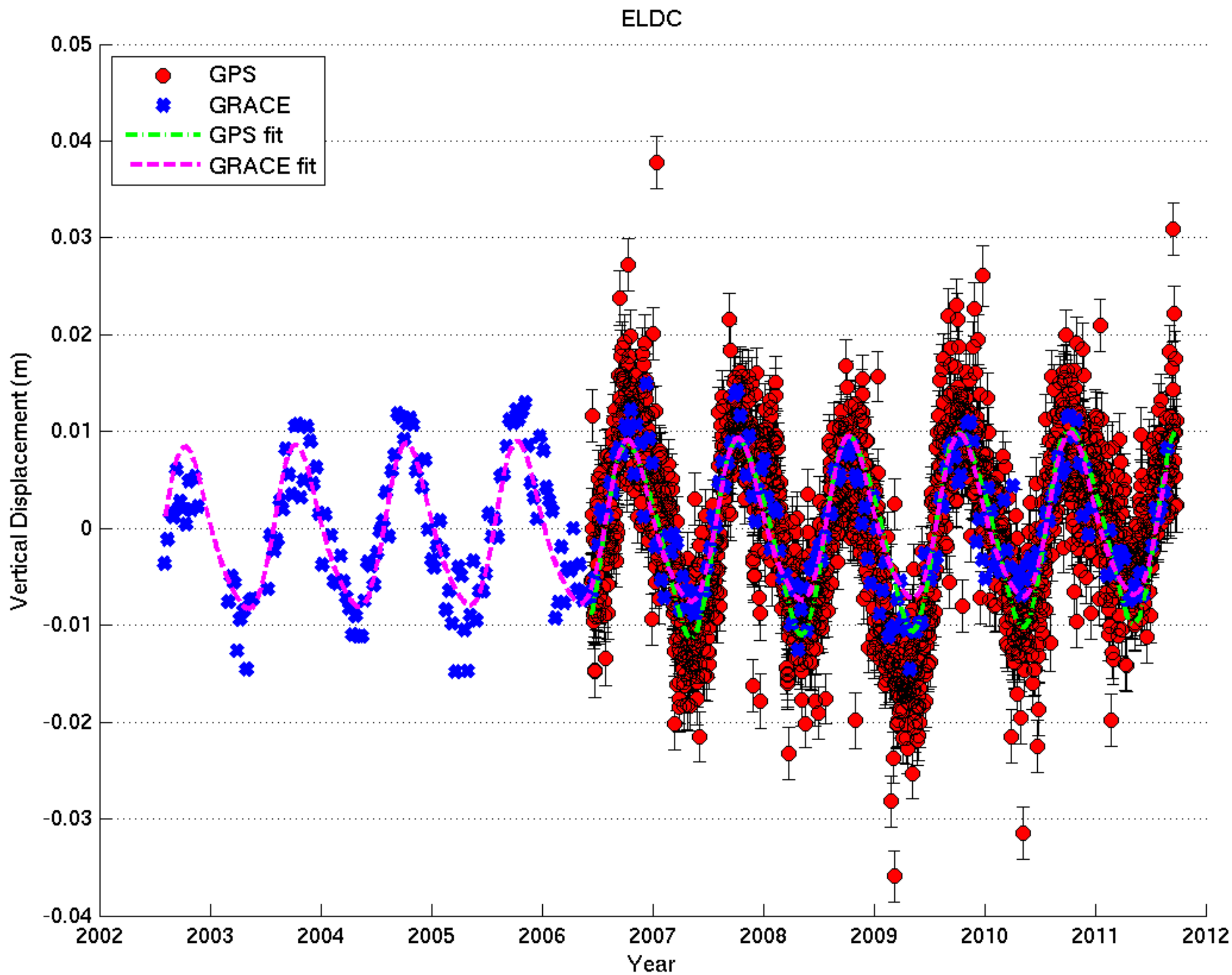


Loading on Seasonal Timescales

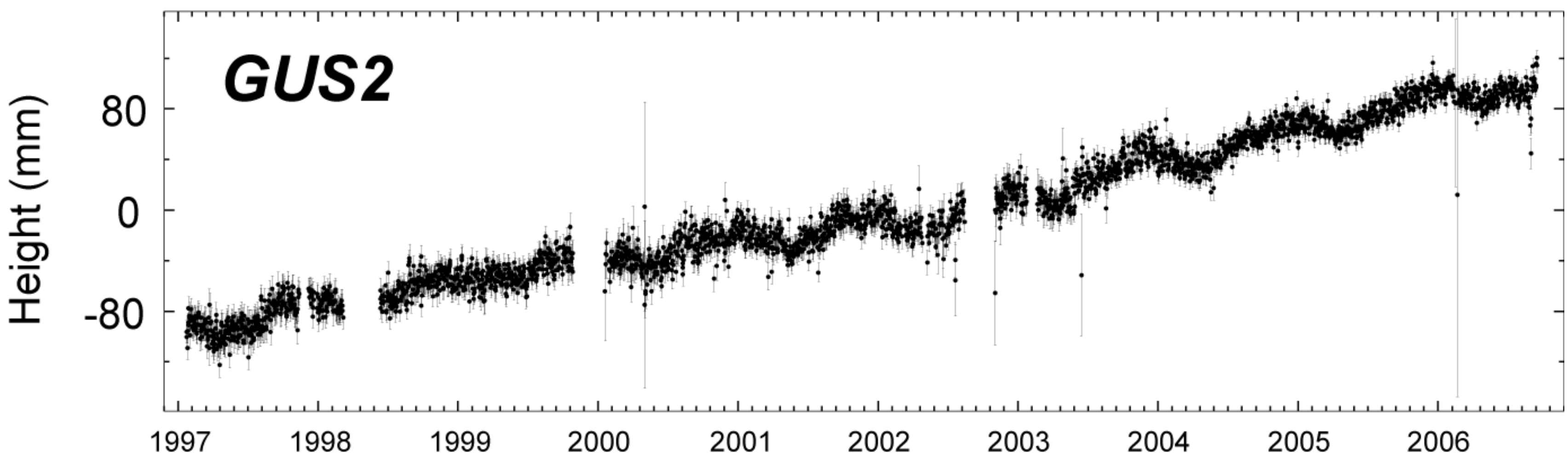


- Dominant control for seasonal/annual periods is hydrology.
- Elastic loading theory can be used to explain signal, using a load model derived from GRACE gravity change.

Comparing GPS and GRACE



Isolating Seasonal Height Variation

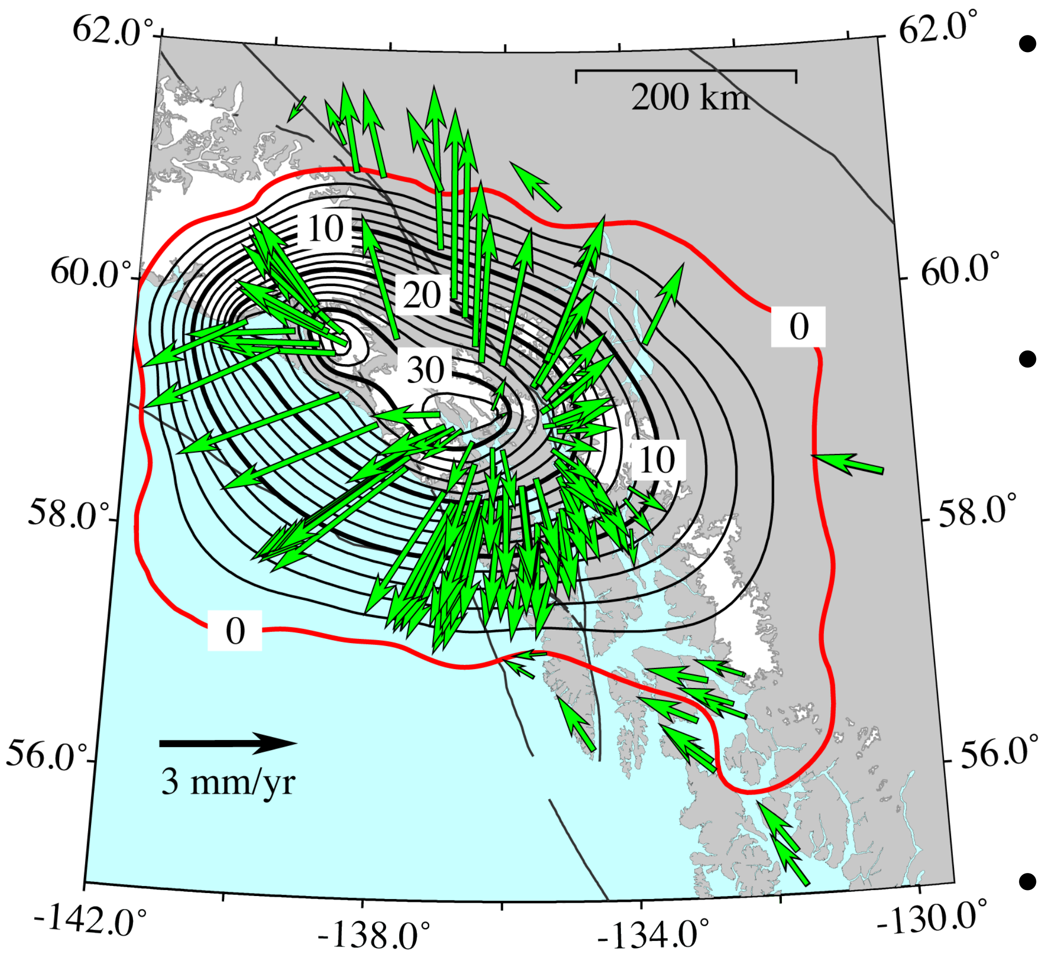


1. Remove linear or long-term trend
2. Stack residuals by fractional year (e.g. Jan 1-10), to get mean residual for that time period
   1. 40 seasonal bins (9.125 days each)
   2. 5 point smoothing applied to bin averages
   3. Daily seasonal variation derived by linear interpolation

Horizontal Displacements

- Horizontal displacements can be computed by the same methods as the vertical
- Horizontal displacements are usually 1/5 to 1/10 as large as the vertical
- Horizontals very sensitive to details of viscosity structure and especially 3D variations
  - Various people's horizontal predictions generally don't resemble each other

Glacial Unloading



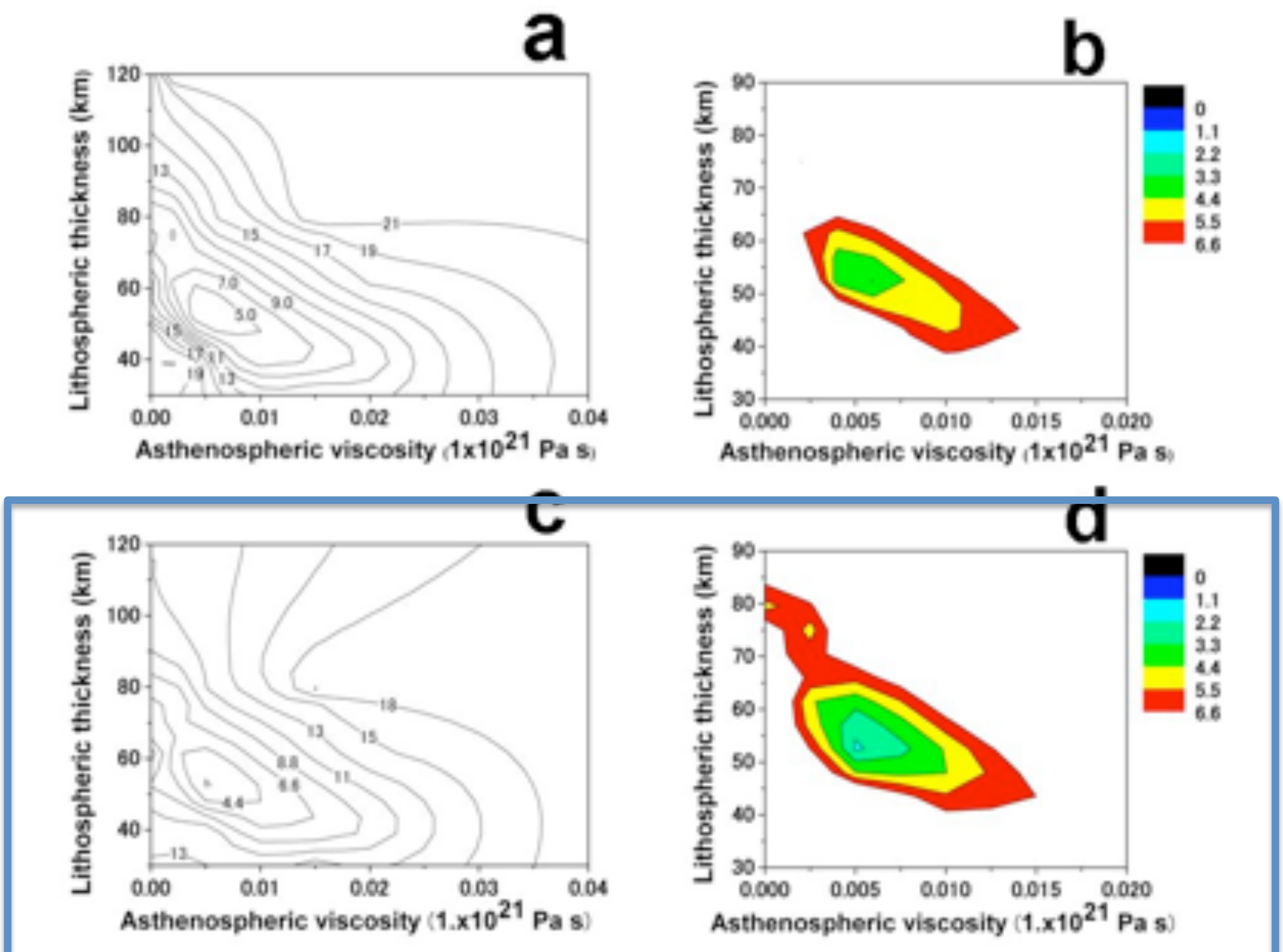
Elliott et al. (in press, JGR)

- Uplift in SE Alaska caused by post-Little Ice Age glacier retreat exceeds 30 mm/yr
- Load history is known; earth model adjusted to fit vertical velocities
  - 50 km elastic lithosphere
  - 3.7×10^{18} Pa s viscosity asthenosphere, 110 km thick
- Horizontal predictions from TABOO shown

GIA Modeling

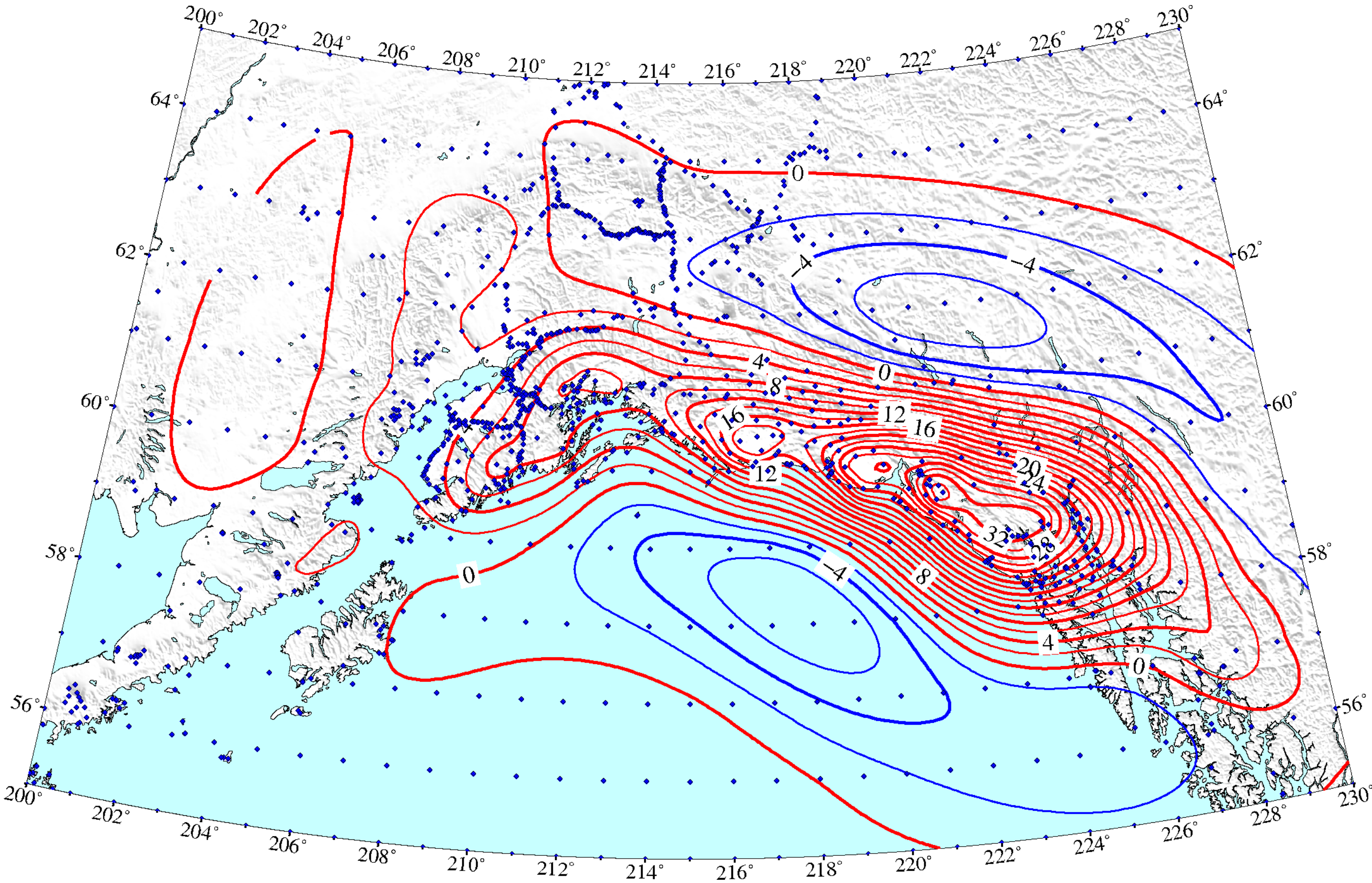
- Uplift rates can be predicted given a load history and the elastic and viscosity structure of the Earth.
  - Calculations based on global, viscoelastic loading theory. Load is represented as a series of disks.
- Combination of instantaneous (elastic) response, time-delayed viscous response
- When we have uplift and load, we can search for the earth model that best explains the data.
- Key parameters: Lithospheric thickness, asthenospheric viscosity and thickness

Earth Model

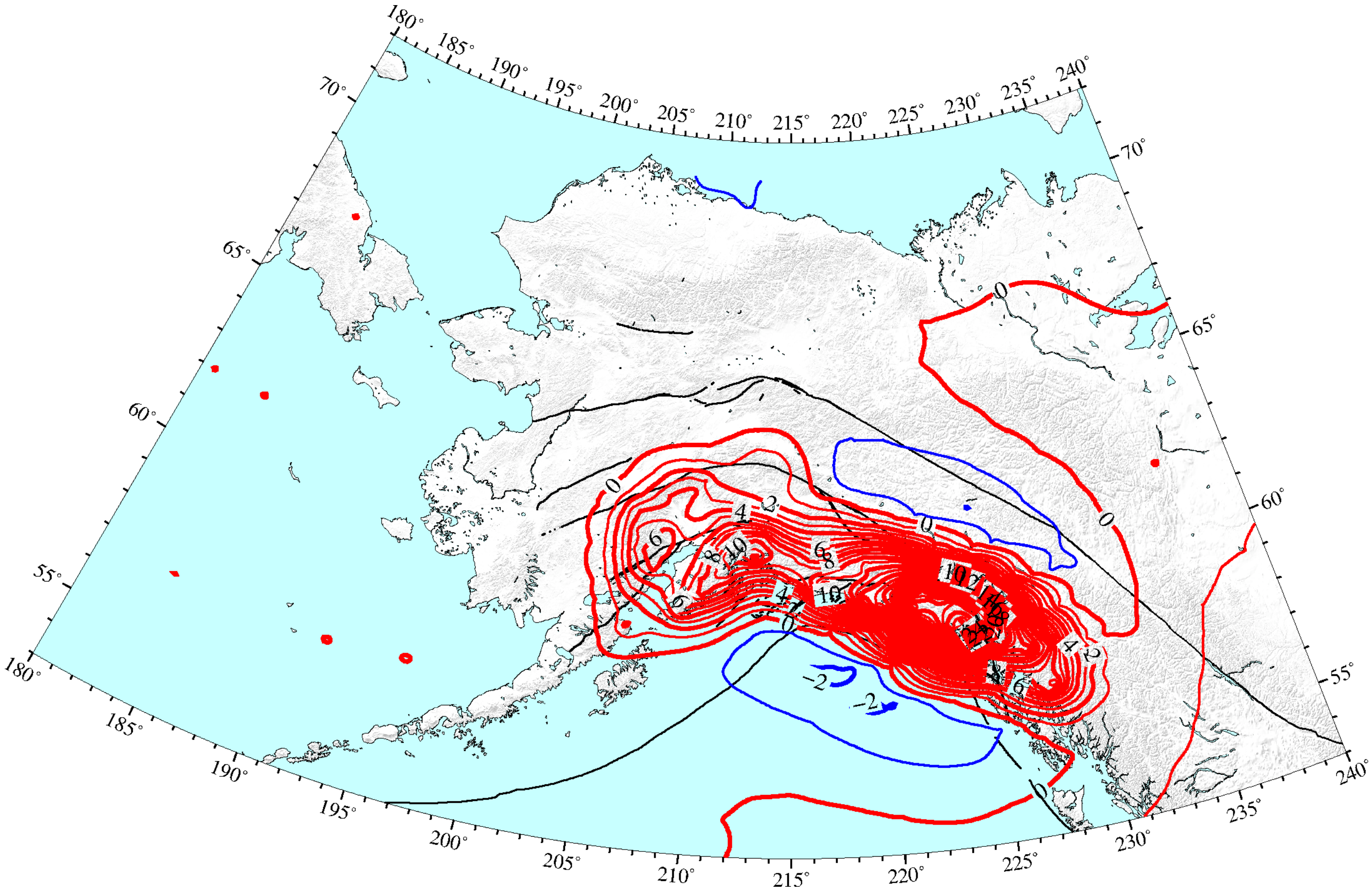


Sato et al. (in press)

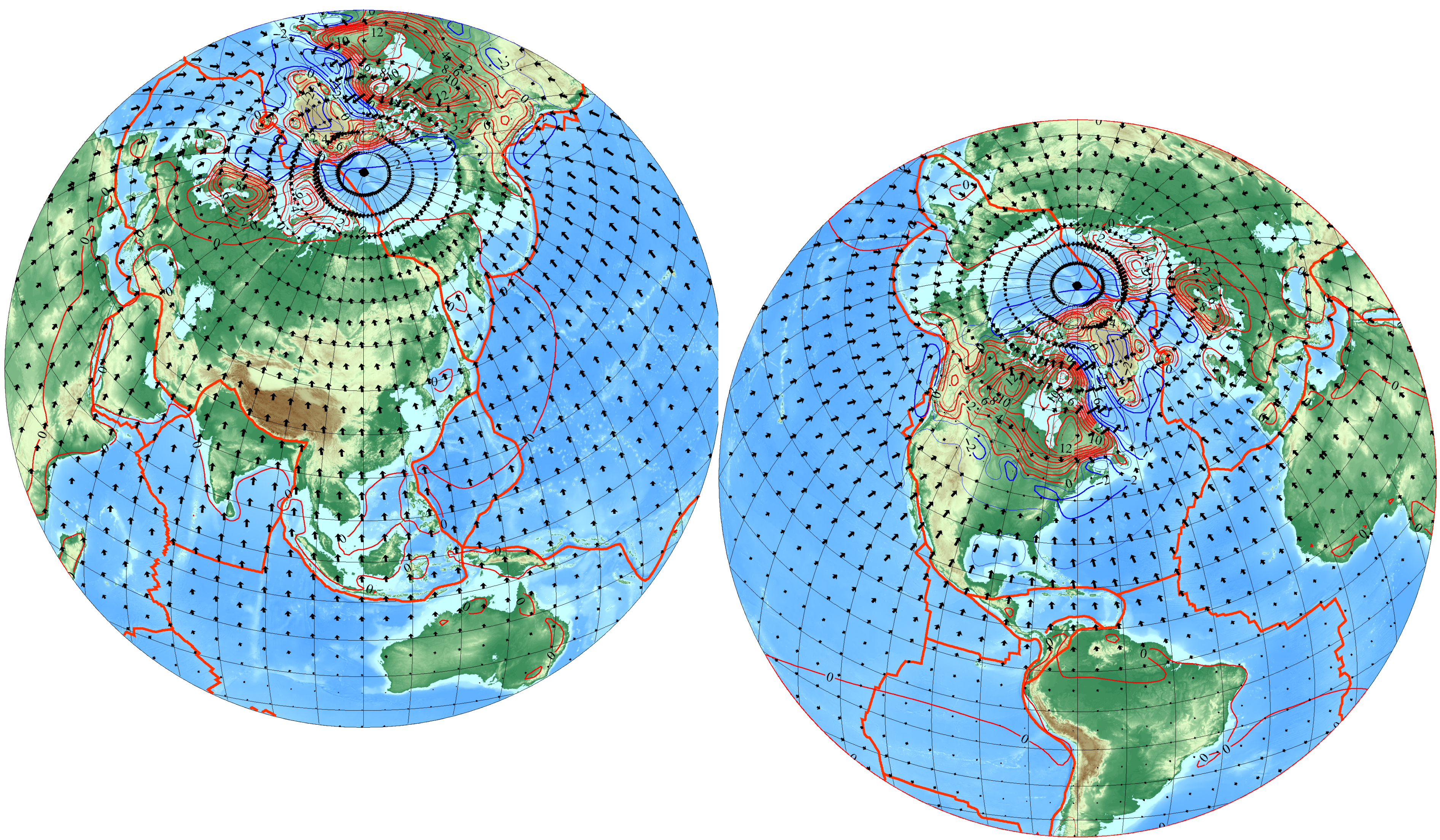
Predicted Uplift/Subsidence Rates



Predicted Uplift/Subsidence Rates



ICE-6G



ICE-5G vs ICE-6G

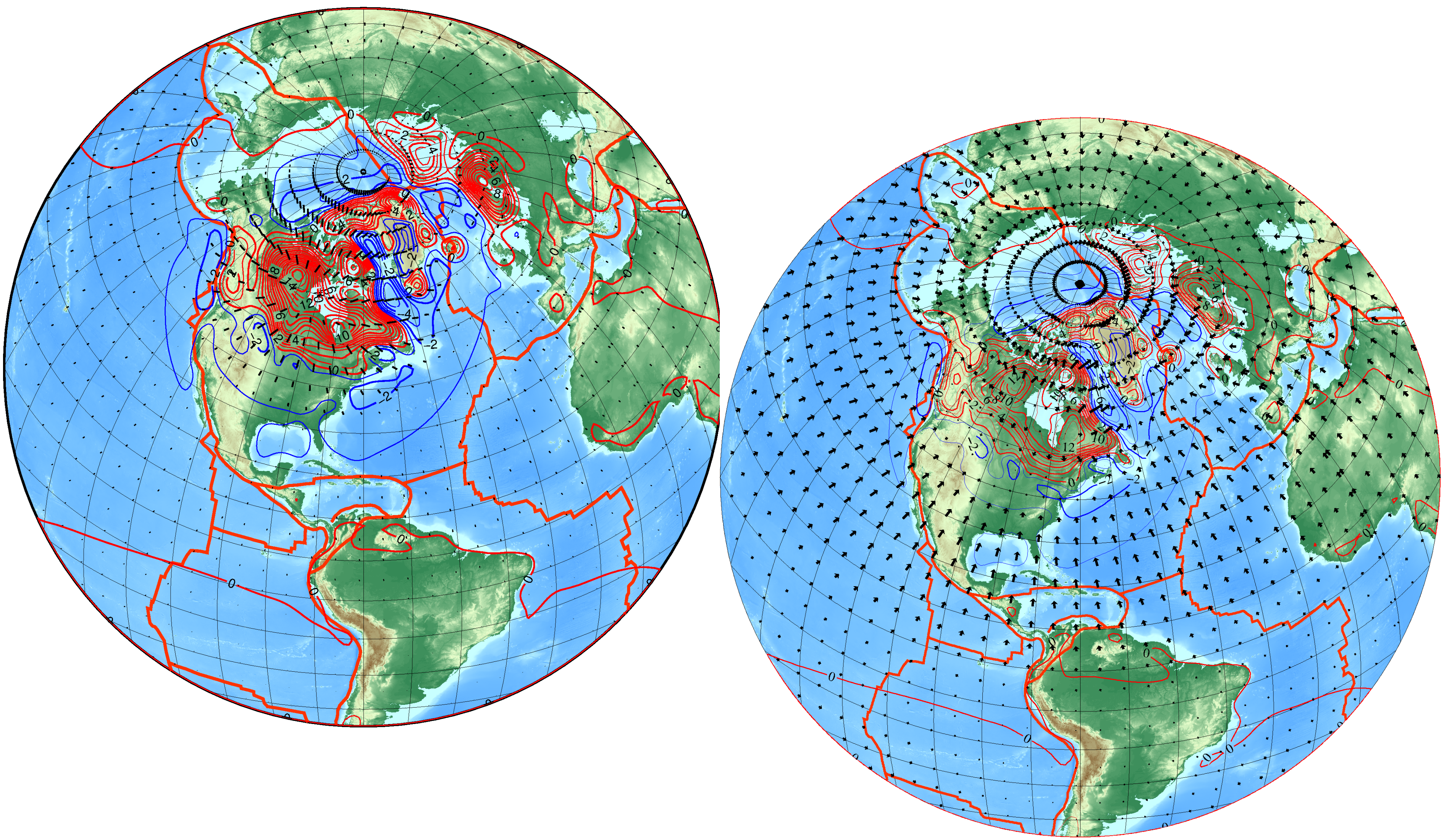
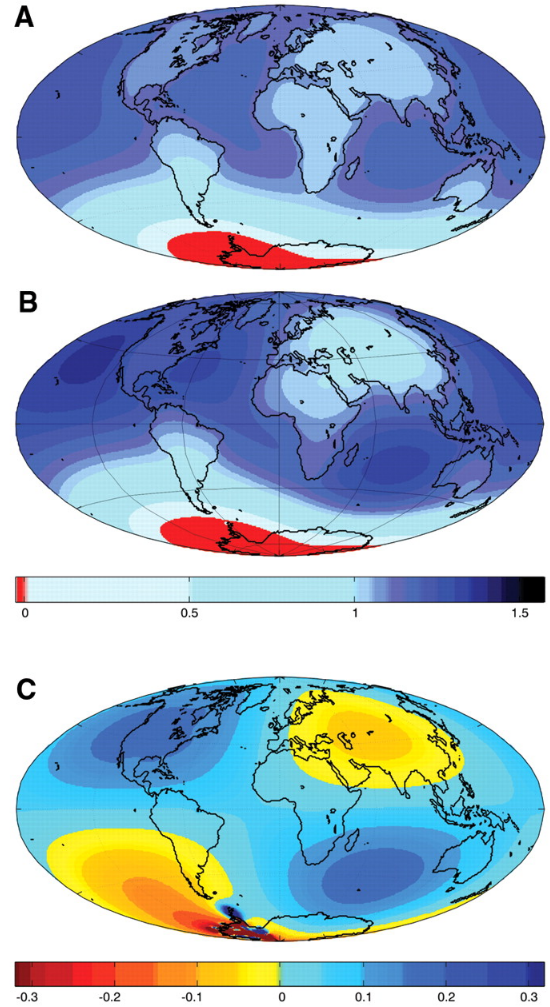


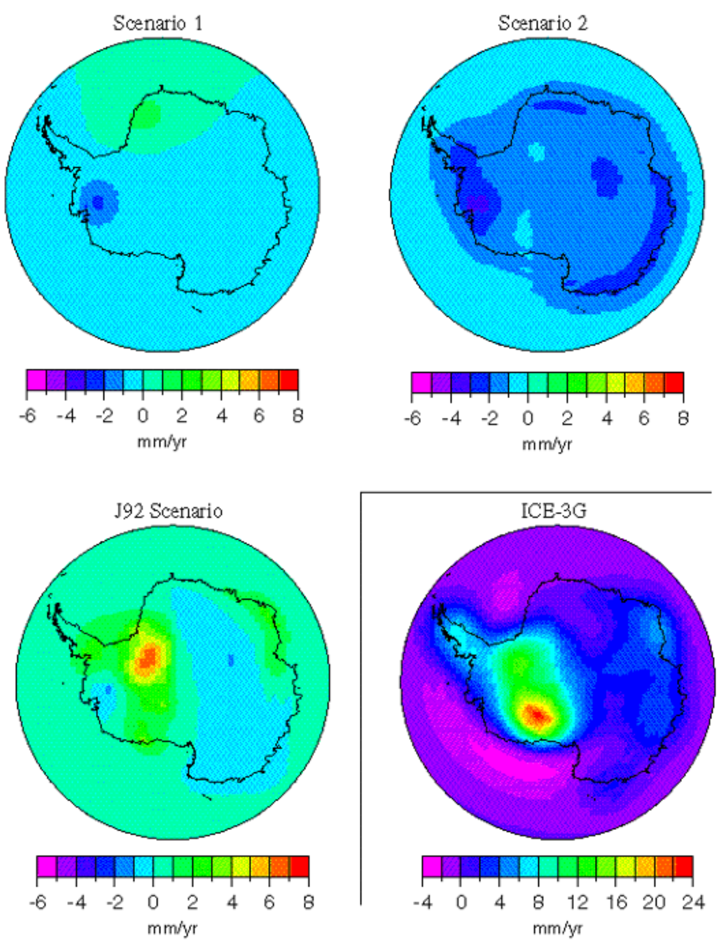
Fig. 1. Sea-level change in response to the collapse of the WAIS computed by using (A) a standard sea-level theory (5), which assumes a nonrotating Earth, no marine-based ice, and shorelines that remain fixed to the present-day geometry with time, as well as (B) a prediction based on a theory (6) that overcomes these limitations



J. X. Mitrovica et al., Science 323, 753 (2009)

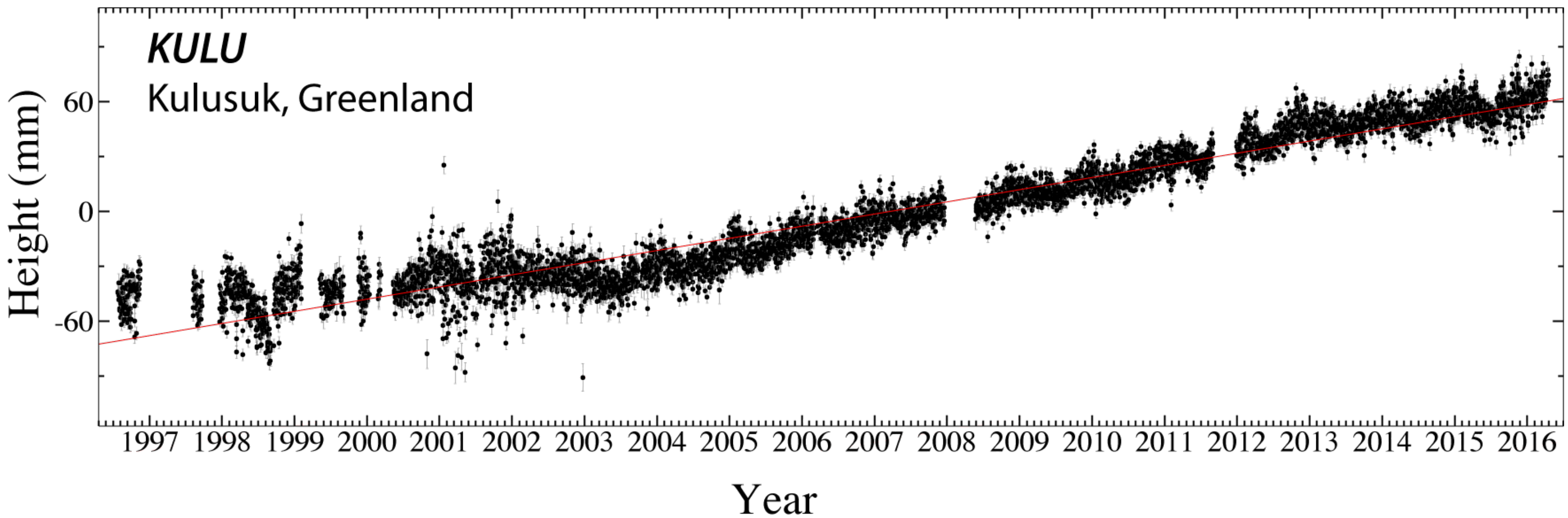
Published by AAAS

How Can We Measure Polar Ice Changes?



- As in Alaska, as the Greenland and Antarctica ice sheets begin to melt, we will observe uplift
  - Present-day: elastic response
  - LGM: late stage viscous response
- Greenland: LGM response is small
- Antarctica: LGM response may be significant

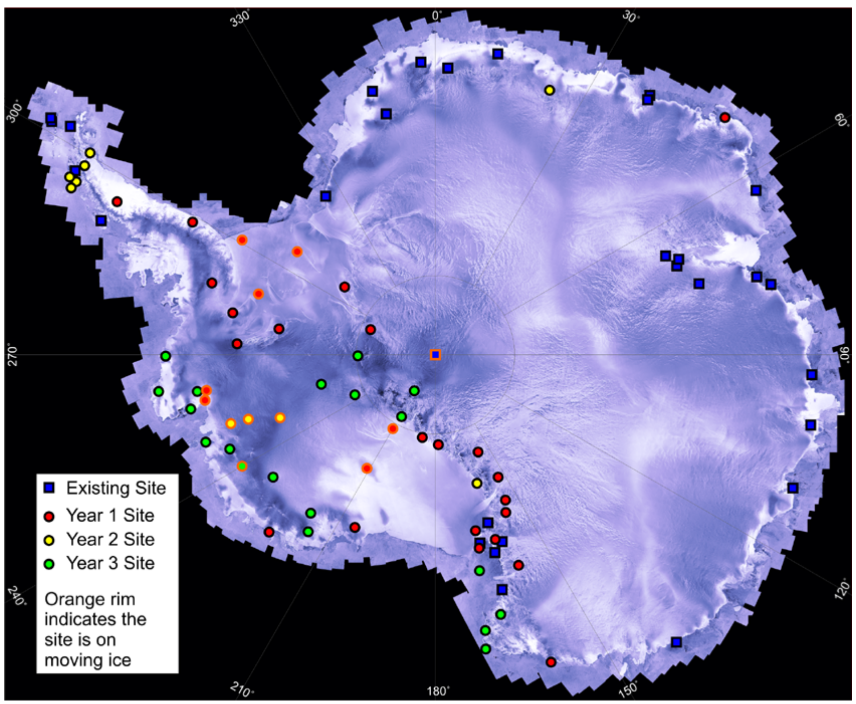
An Example Site: Kulusuk

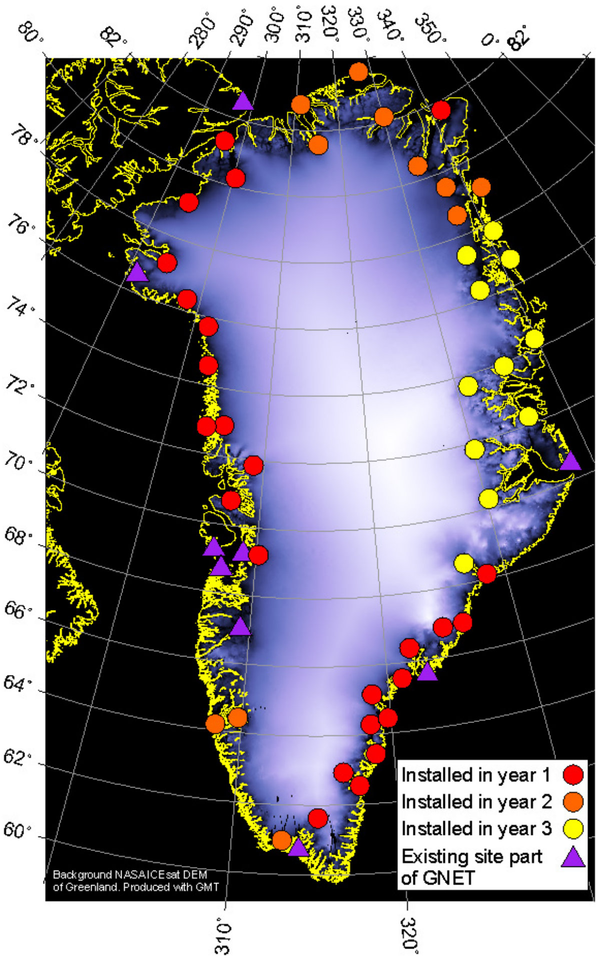


- Kulusuk in SE Greenland
- Before 2003, no vertical motion or perhaps slight subsidence
- Significant uplift after 2003

POLENET Sites

POLENET





Legendre Polynomials

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

Legendre Polynomials

- They appear in a variety of problems because of this relation, where γ is the angle between the vectors **x** and **x'**

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_\ell(\cos\gamma)$$

- Legendre polynomials are orthogonal over the domain -1 to 1:

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{nm}$$

Spherical Harmonics

- Spherical harmonics are synthesized from the Associated Legendre functions, which can be derived from the Legendre polynomials by:

$$P_{nm}(\theta) = \sin^m \theta \frac{d^m}{d(\cos\theta)^m} P_n(\cos\theta)$$

$$P_{nm}(x) = \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

- The (*n*,*m*)th spherical harmonic is

$$S_{nm} = \left(C_{nm} \cos m\phi + S_{nm} \sin m\phi\right) P_{nm}(\cos\theta)$$

Spherical Harmonics

- These functions are orthogonal over a domain corresponding to the surface of a sphere

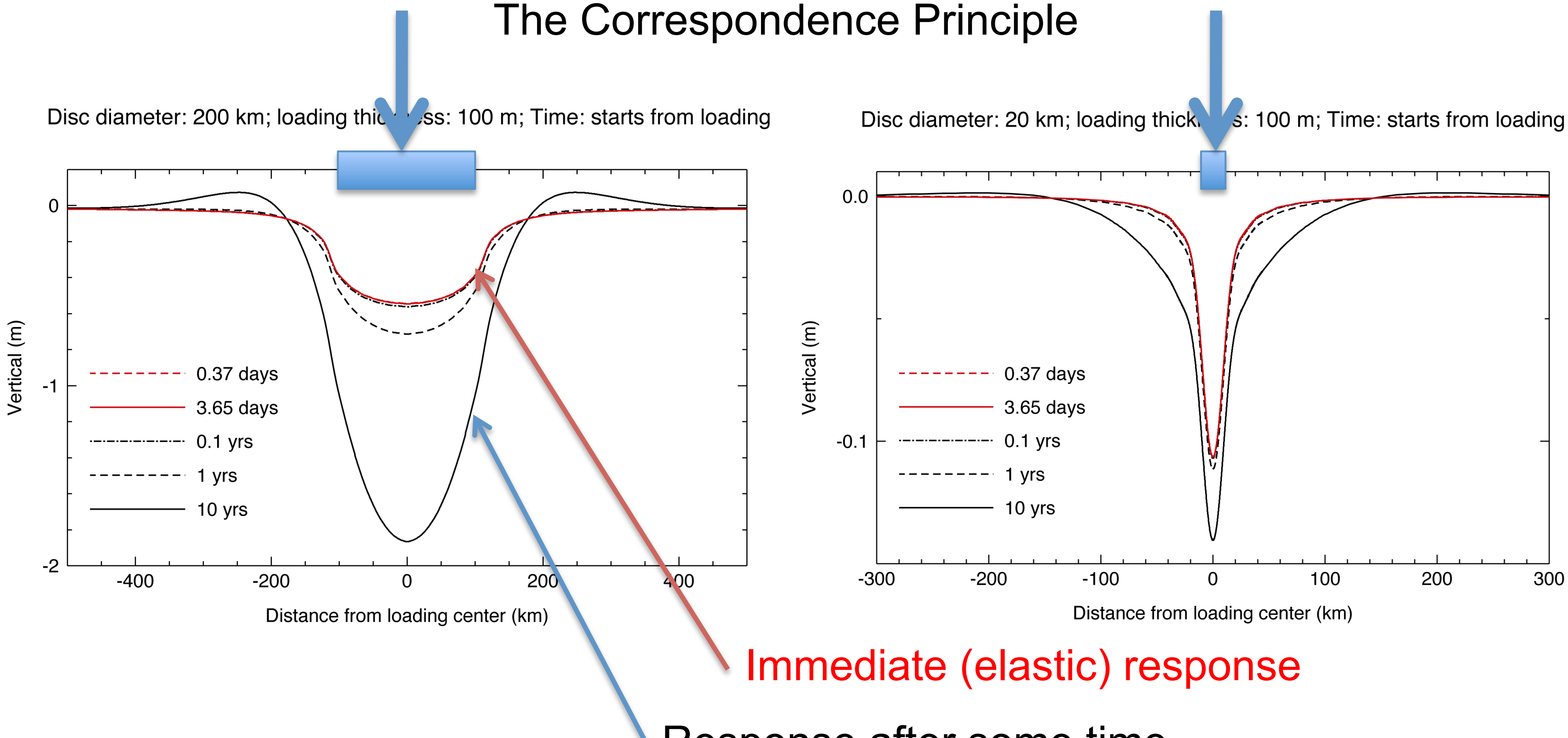
$$\int_0^{2\pi}\int_0^{\pi} S_{nm}(\theta,\phi)S_{pq}(\theta,\phi)\,d\theta d\phi = 0 \quad n \neq p, m \neq q$$

$$\int_0^{2\pi}\int_0^{\pi} S_{nm}(\theta,\phi)S_{pq}(\theta,\phi)\,d\theta d\phi = \frac{4\pi(n+m)!}{(2-\delta_{0m})(2n+1)(n-m)!} \quad n = p, m = q$$

  - Integration of θ is from 0 to π
  - Integration of ϕ is from 0 to 2π
- Normalized versions of the spherical harmonics are generally used so that the inner product of S_{nm} with itself equals 1.

Time Dependent Surface Loading Problems

Love's loading theory (elastic)
+
The Correspondence Principle



Immediate (elastic) response

Response after some time

Computing Loading Deformation

- Two basic methods of computing loading deformation:
  - 1. Greens function method
    - Compute deformation due to a point load or a load of specific shape, where load has unit magnitude
    - Convolve the actual spatial load with the Greens function
  - 2. Love's loading theory
    - Represent the load in terms of spherical harmonic functions
    - Computation is easy with tabulated ***Love Numbers***, which depend on the earth model
- In practice, the Greens functions are often computed using Love's theory.

Greens Functions Review

- The response is a convolution of the load with the Greens function for response to a (point) load. In 1D, it would look like this:

$$d(x) = \int G(x - x')q(x')dx'$$

  - (response) = integral over range of (response to point load at x')(amount of load at location x')

- If we integrate over an area A

$$d(x) = \iint_A G(x - x')q(x')dA$$

  - **x** and **x**' are now vectors, and dA is the differential of area.

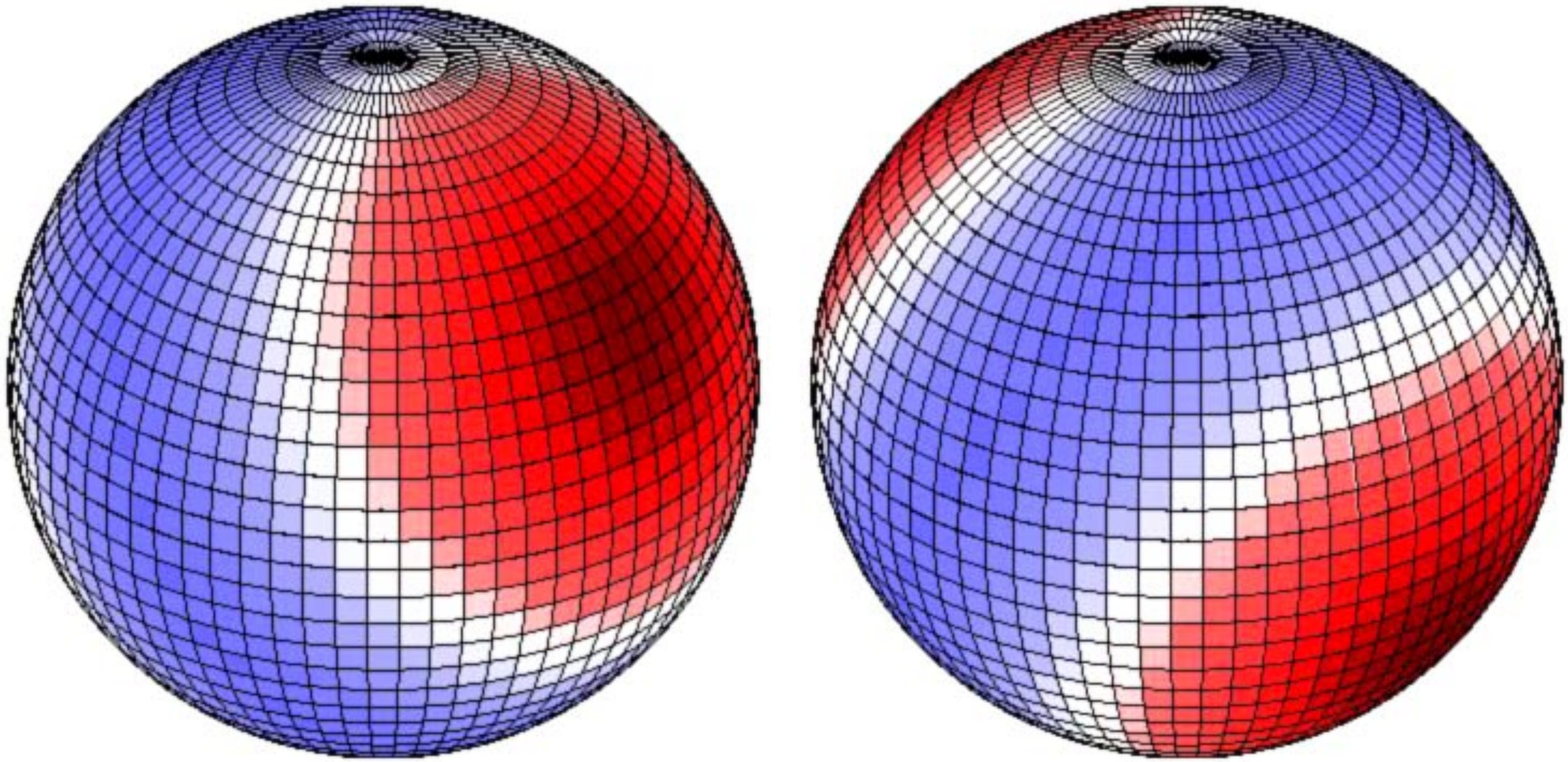
Greens Functions for Non-Point Loads

- The previous equations are for point loads.
  - Requires knowing load as a continuous function
  - May want to approximate load with a gridded version
  - Sum up response to each load/gridpoint
- The response for a disk load of radius R is commonly used.
- Rectangular loads may be used with half-space problems
- Greens functions pre-computed, calculation for reasonably complex loads is fast and simple.
  - But how to compute the Greens function in the first place?

General Elastic Loading Theory

- Elastic first; viscoelastic using ***correspondence principle***.
- Insight comes from tides – described in terms of a tidal potential
- Loading and external forces (tides) apply forces to deformable earth. Theories have similar form
- Shape of earth linked to gravitational potential
  – Can describe any force in terms of a ***deforming potential***

Tidal Potential



Analogy to Tides

- The gravitational potential including the effects of tides is $V(t) = V_{static} + \Delta V(t)$
- Equipotential surface is a surface of equal gravitational potential
  - Shape defines "downhill"
- The ***geoid*** is the equipotential surface (for the V_{static} term only, $V = V_0$) that corresponds to mean sea level.
- Equipotential surfaces will be displaced as the tidal potential $\Delta V(t)$ changes
  - Displacement of equipotential $\Delta N = -\Delta V/g$
- Fluid conforms shape to equipotential surface

Response of Fluid

- Fluids can't maintain "topography", flow to equalize pressure
  - Fluid flows from high gravitational potential to lower potential (== "downhill")
  - In real ocean, this flow takes time, so tides are not a perfect match to potential because of tidal currents
    - Tidal modeling involves solving fluid flow equations subject to changing potential.
    - We'll ignore these details.

Response of Solid Earth to Tides

- Not a fluid, but also deforms. Define the number h as the ratio of (change in height of solid earth) to (change in height of ideal fluid), in response to a change in potential ΔV.
- Now describe the response of solid earth to the tides by solving the tidal problem and using the proportional response (h, ***Love number***).
- Change in height of solid earth = $-(h/g)\Delta V$

Tidal Love Numbers

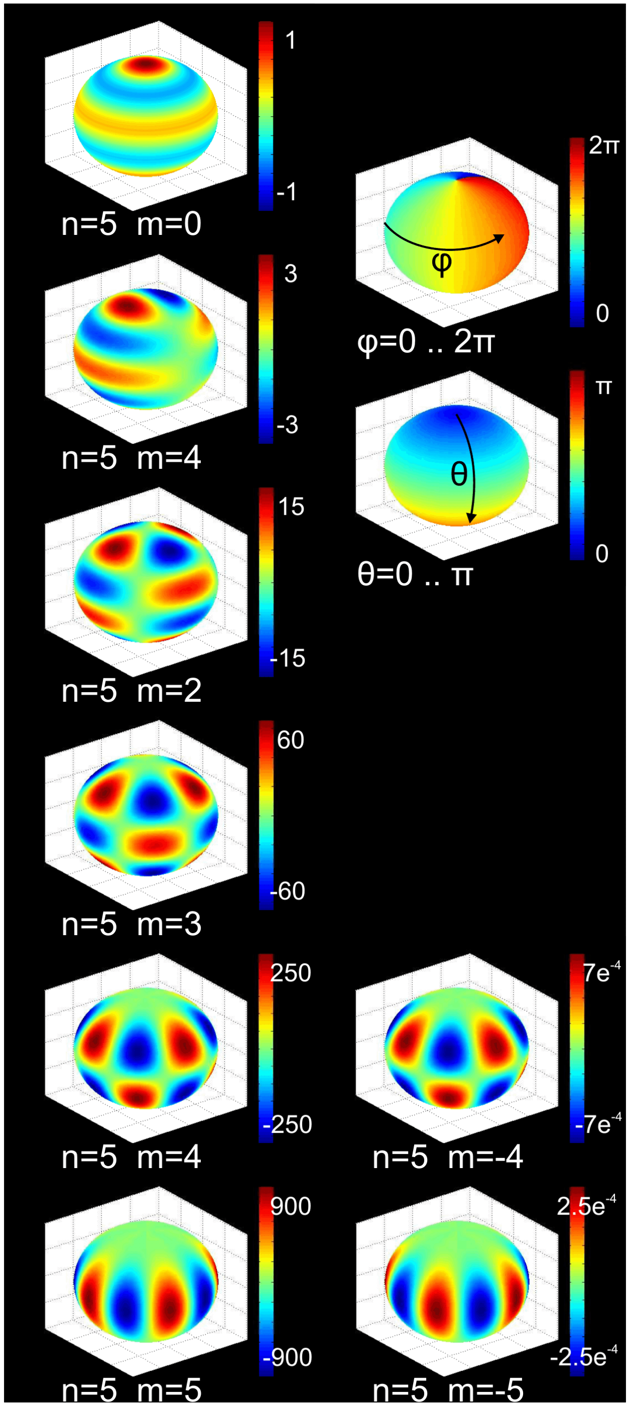
- The Love number h depends on the length scale of the load/potential change
  - Natural to use spherical harmonics to represent load potential, because they are the natural basis for gravitational potential on a sphere
  - There is one Love number for each spherical harmonic degree n.
- The Love theory says the response is proportional to magnitude of the tidal potential, but with a different constant of proportionality for each wavelength

Spherical Harmonics

- One common notation for the gravitational potential is:

$$V = \frac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n \left(C_{nm}\cos m\varphi + S_{nm}\sin m\varphi\right)P_{nm}(\cos\theta)$$

  - θ is co-latitude (0 at pole, $\pi/2$ at Equator)
  - ϕ is longitude, R_e radius of earth
- Pnm are the associated Legendre polynomials, derived from the Legendre polynomials Pn. These are a set of orthogonal polynomials.
- The harmonics are orthogonal, defined by integration over the surface of a sphere.
- All spherical harmonics solve Laplace's equation



Example Harmonics

- Slightly different normalization conventions are used in different fields.
  - You have to be sure to use the normalization coefficients and orthogonality relationship for the same set
- Geomagnetism: Schmidt
- Gravity/Geodesy:

Spherical Harmonics

- In geodesy, a slightly different notation is common:

$$Y_{inm} = P_{nm}(\cos\theta)\begin{Bmatrix}\cos m\varphi & i=1 \\ \sin m\varphi & i=2\end{Bmatrix}$$

$$C_{inm} = \begin{Bmatrix}C_{nm} & i=1 \\ S_{nm} & i=2\end{Bmatrix}$$

- So $$V = \frac{GM}{r}\sum_{i=1}^{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n C_{inm}Y_{inm}$$
- There are other slightly different notations.
- The orthogonality relationship is:

$$\iint_{sphere} Y_{inm}Y_{jpq} = \begin{Bmatrix}\dfrac{4\pi}{\Pi_{nm}^2} & i=j, n=p, m=q \\ 0 & otherwise\end{Bmatrix}$$

Love's Loading Equation

- The loading problem is solved in a similar way as the tidal problem (Munk and McDonald, 1960):
  - Treat the load as a thin shell on surface with surface density $\sigma(\Omega)$ [Ω stands for (θ,ϕ)]

$$\sigma(\Omega) = \sum_{i=1}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sigma_{inm} Y_{inm}(\Omega)$$

  - What happened to the n=0 terms? They must be zero by conservation of mass
  - Load represents fluids that could be redistributed across surface

Load Potential

- Define a perturbing potential or load potential

$$W(\Omega) = \sum_{n=0}^{\infty} W_n(\Omega) = \frac{4\pi R_e g}{M_e} \sum_{n=0}^{\infty} \sum_{i=1}^{2} \sum_{m=0}^{n} \frac{\sigma_{inm} Y_{inm}(\Omega)}{(2n+1)}$$

- Vertical displacement is

$$\Delta u_h(\Omega) = \sum_{n=0}^{\infty} \frac{h_n' W_n(\Omega)}{g}$$

- Lateral (horizontal displacement is)

$$\Delta u_l(\Omega) = \sum_{n=0}^{\infty} \frac{l_n' \left(\hat{l} \cdot \nabla W_n(\Omega)\right)}{g}$$

Load Love Numbers

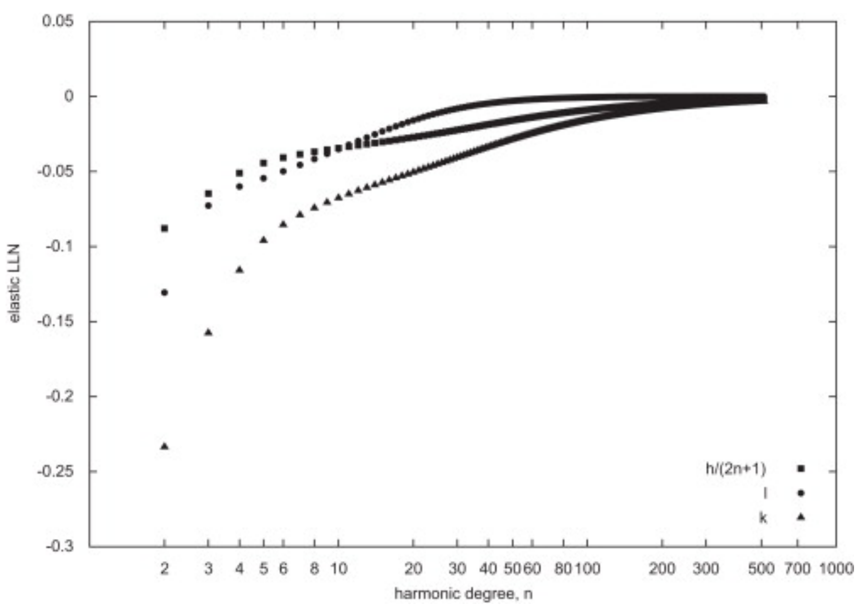
- The ***Load Love Numbers*** (h_n', l_n', k_n') are computed for a given earth model. The k_n' are for gravitational potential changes.
  - Complexities result because the load Love numbers are reference frame dependent (center of mass of solid earth vs CM of earth system, etc). Below are in CM of Earth System frame:

# degree #	h'	l'	k'
0	0.0000000000D+00	0.0000000000D+00	-1.0000000000D+00
1	-0.1285877758D+01	-0.8960817937D+00	-0.1000000000D+01
2	-0.9915810331D+00	0.2353293958D-01	-0.3054020195D+00
3	-0.1050767745D+01	0.7014846821D-01	-0.1960294041D+00
4	-0.1053393012D+01	0.5888944962D-01	-0.1336652689D+00
5	-0.1086317605D+01	0.4635490492D-01	-0.1047066267D+00
6	-0.1143860336D+01	0.3875790076D-01	-0.9033564429D-01
7	-0.1212408459D+01	0.3435761766D-01	-0.8206984804D-01
8	-0.1283943275D+01	0.3162521937D-01	-0.7655494644D-01
9	-0.1354734845D+01	0.2976799724D-01	-0.7243844815D-01

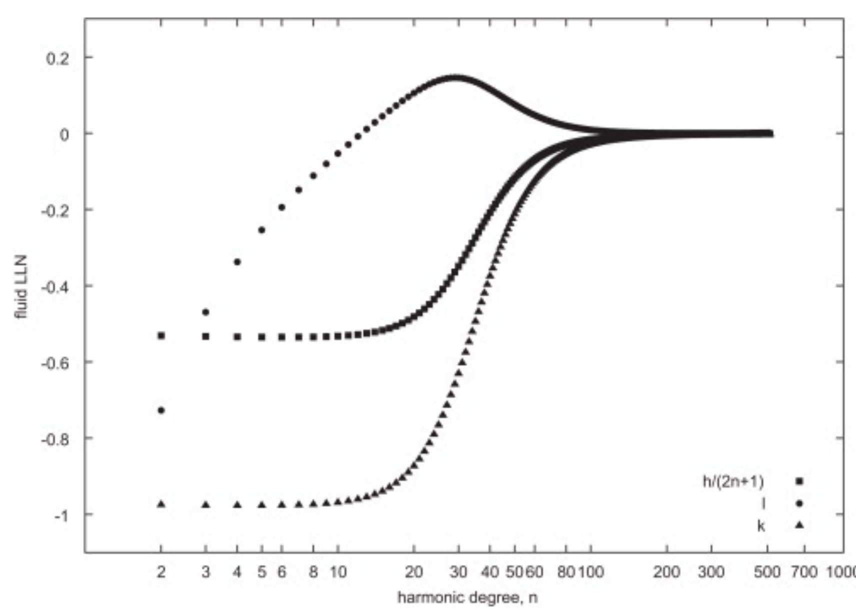
Higher Order Love Numbers

The Center of Mass of Solid Earth (CE) frame is a natural one to use for loading problems. The figures below come from Spada (2008)

Elastic



Fluid (fully relaxed)



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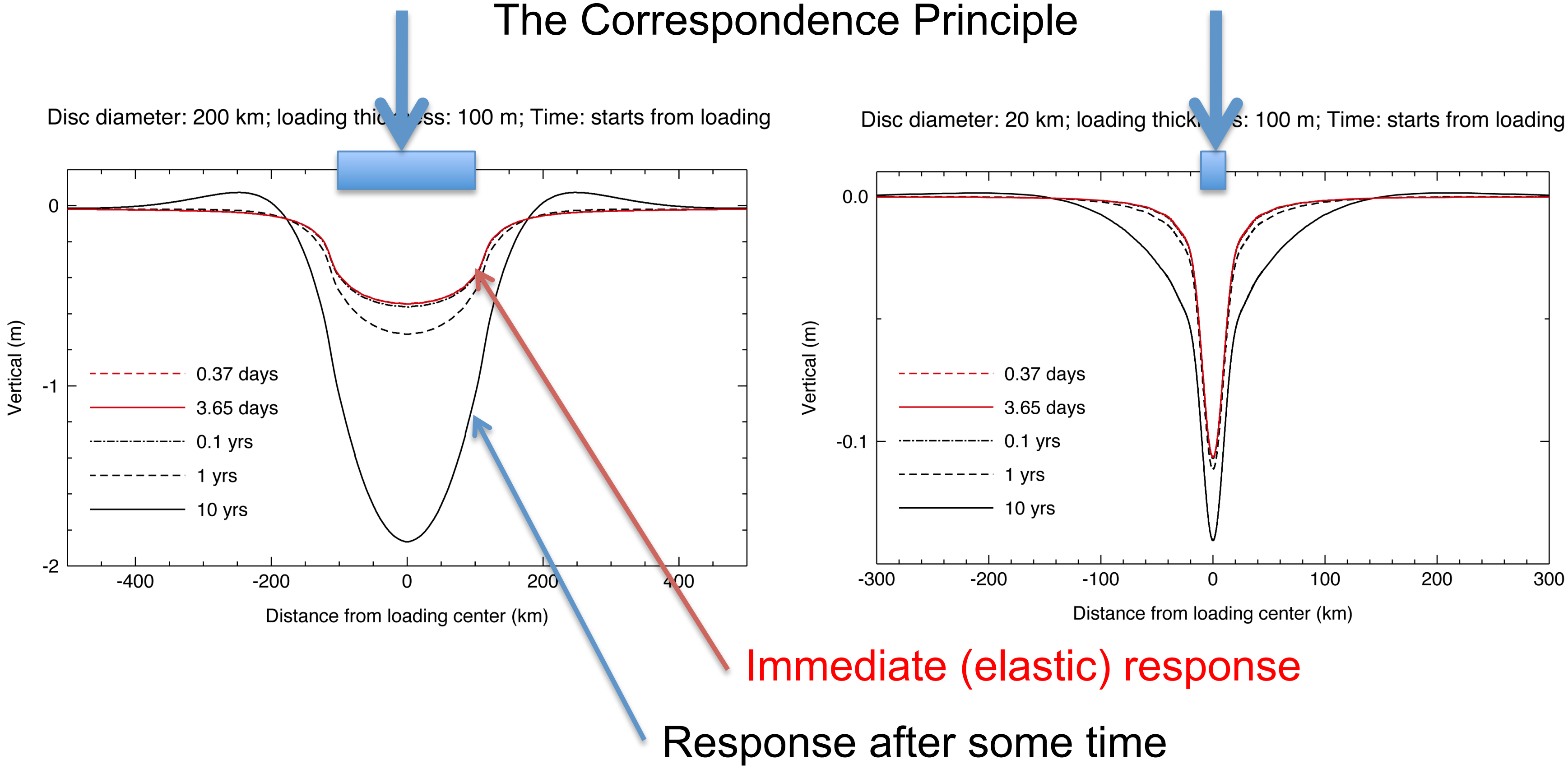


Correspondence Principle and Viscoelasticity

- Constitutive equations for an elastic material (Hooke's Law): $\sigma = 2G\varepsilon$ or $\varepsilon = \sigma/2G$
- For a (Maxwell) viscoelastic material we have:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{2G} + \frac{\sigma}{2\eta}$$

- Strain rate depends on both the shear modulus and the viscosity
- Now apply the Laplace transform

Laplace Transform and Viscoelasticity

- The Laplace transform is an integral transform

$$LT\left[f(t)\right] = \hat{f}(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

- The inverse transform

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \hat{f}(s)e^{st}ds$$

- There is a one to one correspondence between the Laplace transform pair $f(t)$ $\hat{f}(s)$
  - That means if you can solve a problem in the Laplace transformed domain, you can get the solution in the original (time) domain via the inverse transform

Laplace Transform

- The Laplace transform converts differential equations or integral equations into algebraic equations

$$LT\left[\frac{d}{dt}f(t)\right] = s\hat{f}(s) - f(0)$$

$$LT\left[\int_0^t f(q)dq\right] = \frac{1}{s}\hat{f}(s)$$

Correspondence Principle

- The correspondence principle tells us that the Laplace transform of a viscoelastic problem is equivalent to an elastic problem.
- Compare viscoelastic problem to elastic problem:
  - Equilibrium equations are the same
  - Kinematic (strain-displacement) equations are the same
  - Only the constitutive (stress-strain) equations are different.

Correspondence Principle

- Apply the Laplace transform to the constitutive relation for a Maxwell viscoelastic material, with ε=0 and σ=0 at t=0:

$$LT[\dot{\varepsilon}] = LT\left[\frac{\dot{\sigma}}{2G} + \frac{\sigma}{2\eta}\right]$$

$$s\hat{\varepsilon} = \frac{s\hat{\sigma}}{2G} + \frac{\hat{\sigma}}{2\eta} = \left(\frac{s}{2G} + \frac{1}{2\eta}\right)\hat{\sigma} = \frac{\hat{\sigma}}{2\hat{G}}$$

- The Laplace transformed constitutive relation looks like an elastic solution for $\hat{G} = \frac{Gs}{s + G/\eta}$
- Therefore the inverse Laplace transform of the solution to the elastic problem is a solution to the viscoelastic problem.

Correspondence Principle and Loading

- The loading problem can be solved by computing viscoelastic Love numbers h(s)
  - Like the 1D problem just shown, the dependence on s is only in the constitutive relation, and thus only in the Love numbers.
- After the inverse transform, you simply get a time-dependent Love number $h_n(t)$, and this is the only place the time-dependence is found.
- The time dependence of $h_n(t)$ depends on the time history of the load.

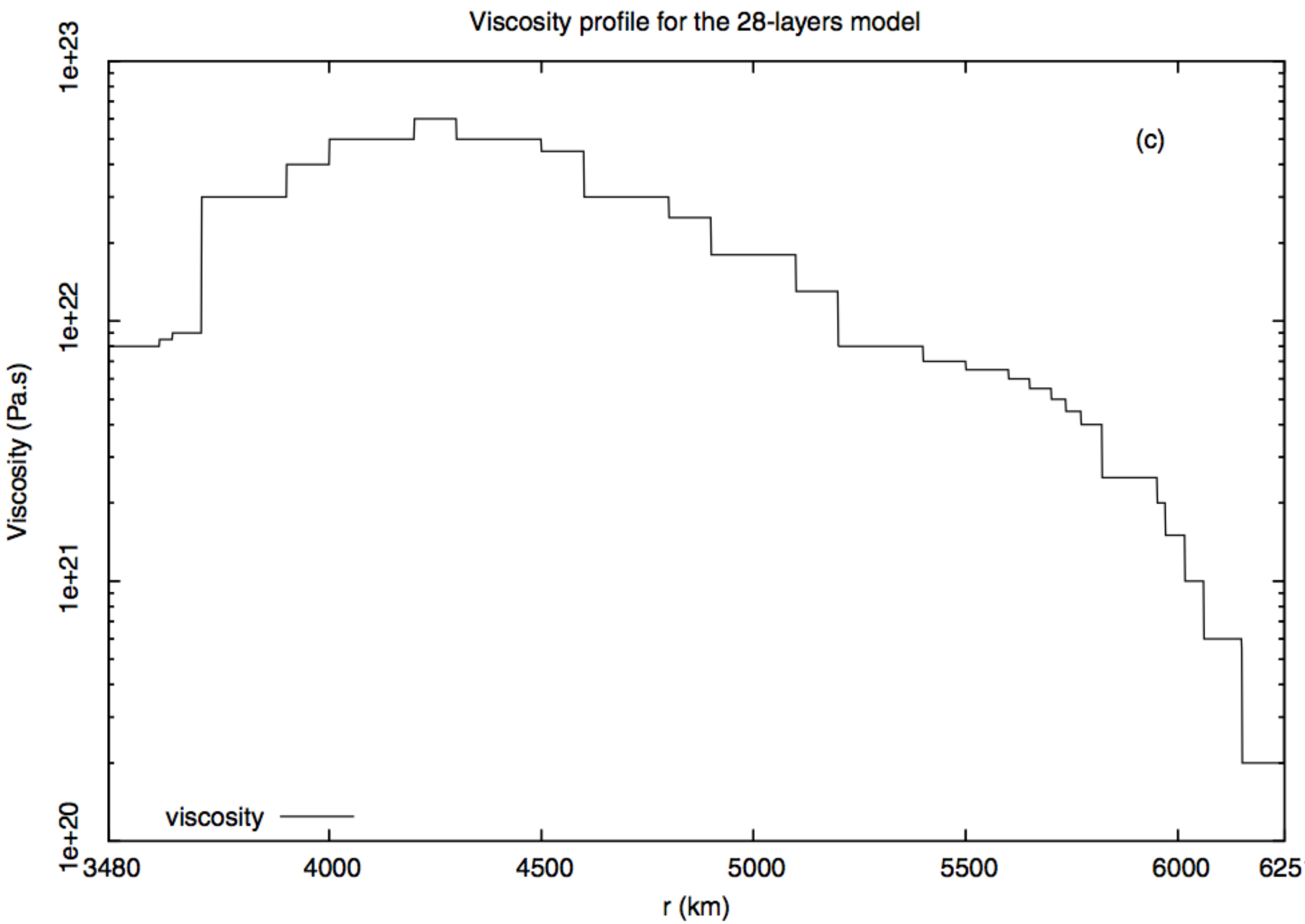
Time History

- For a step function (Heaviside function) load change $\sigma H(t-t_0)$ and a Maxwell viscoelastic material, h(t) will have the form (for t >= t_0):

$$h_n(t) = h_n^E + \tau h_n^V\left(1 - e^{-t/\tau}\right)$$

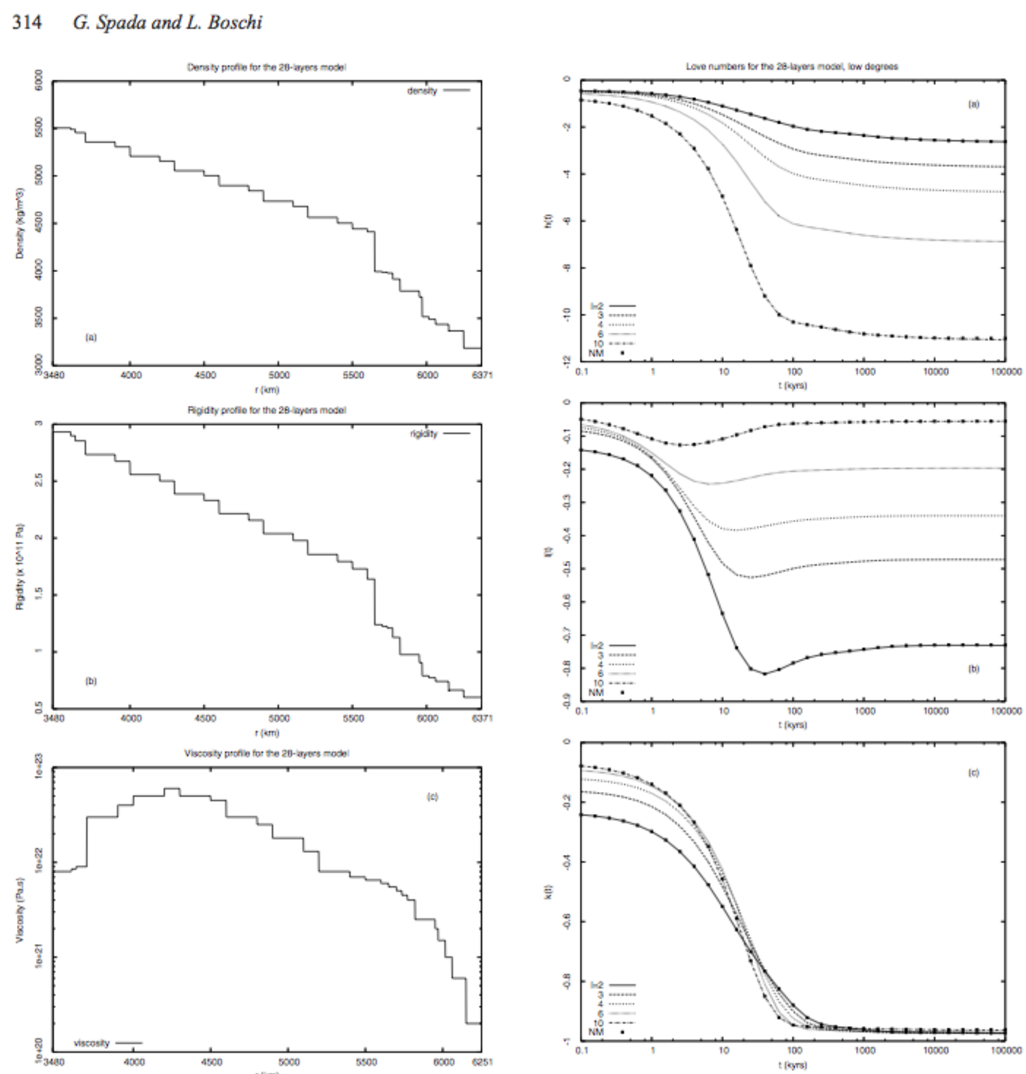
- In this equation, τ is the Maxwell relaxation time and $h_n^E + \tau h_n^V$ is the fully relaxed response (like a fluid), as time goes to infinity
- More complex relations result from layering within the Earth ➔ more relaxation terms

Visualizing h(t)



Spada and Boschi (2006, GJI)

Visualizing h(t)



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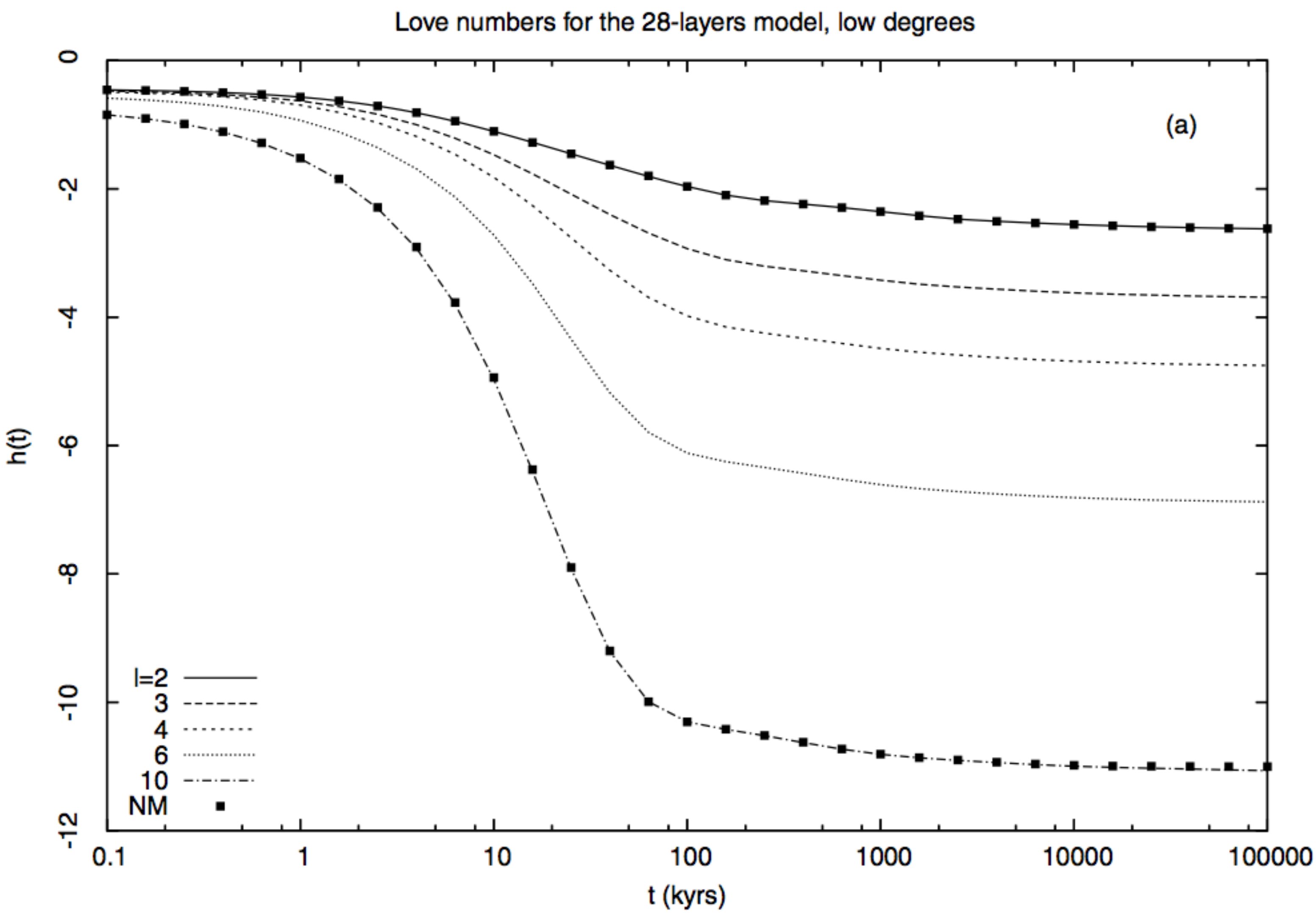
Visualizing h(t)



Spada and Boschi (2006, GJI)

Solving the Surface Loading Problem

Love's loading theory (elastic)
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The Correspondence Principle