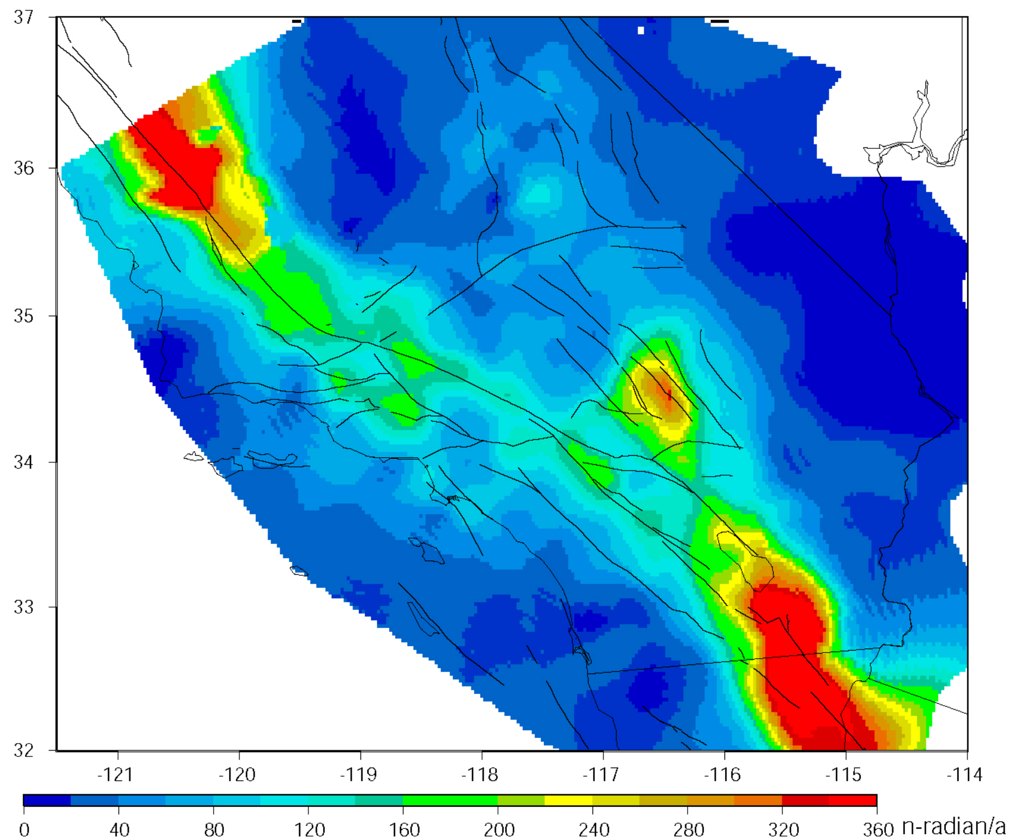
Lecture 14: Strain Examples



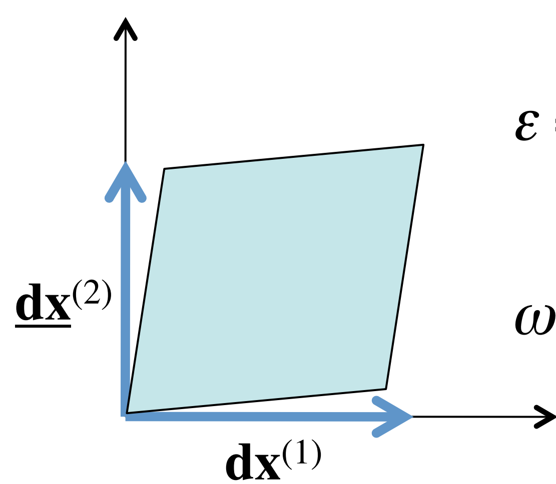
GEOS 655 Tectonic Geodesy

Jeff Freymueller

A Worked Example

$$\Theta_i = e_{ijk}\left(\varepsilon_{km} + \omega_{km}\right) d\hat{x}_j d\hat{x}_m$$

- Consider this case of pure shear deformation, and two vectors $\mathbf{\underline{dx}}_1$ and $\mathbf{\underline{dx}}_2$. How do they rotate?
- We'll look at vector 1 first, and go through each component of Θ.



$$\varepsilon = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega = (0,0,0)$$

$$(1) dx = (1,0,0) = d\hat{x}_1$$

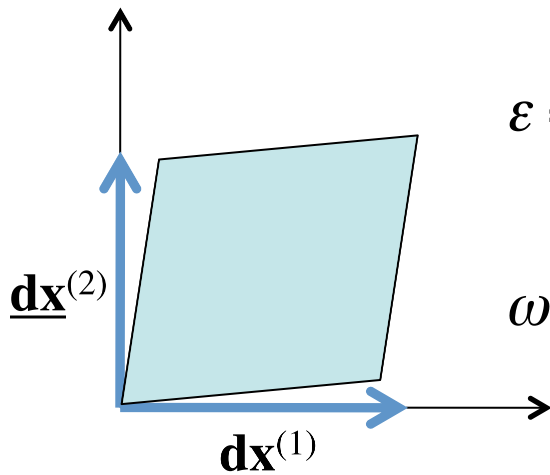
$$(2) dx = (0,1,0) = d\hat{x}_2$$

A Worked Example

- First for i = 1

$$\Theta_1 = e_{1jk}\left(\varepsilon_{km} + \omega_{km}\right)d\hat{x}_j d\hat{x}_m$$

- Rules for e_{1jk}
  - If j or k =1, $e_{1jk} = 0$
  - If j = k =2 or 3, $e_{1jk} = 0$
  - This leaves j=2, k=3 and j=3, k=2
  - Both of these terms will result in zero because
    - j=2,k=3: $\varepsilon_{3m} = 0$
    - j=3,k=2: $dx_3 = 0$
  - True for both vectors



$$\varepsilon = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega = (0,0,0)$$

$$(1)dx = (1,0,0) = d\hat{x}_1$$

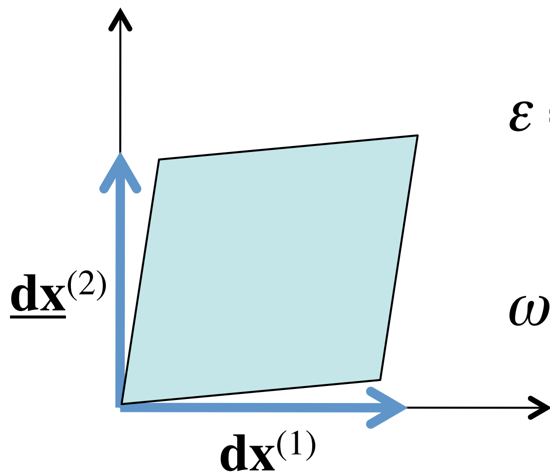
$$(2)dx = (0,1,0) = d\hat{x}_2$$

A Worked Example

- Now for i = 2

$$\Theta_2 = e_{2jk}\left(\varepsilon_{km} + \omega_{km}\right)d\hat{x}_j d\hat{x}_m$$

- Rules for e_{2jk}
  - If j or k =2, $e_{2jk} = 0$
  - If j = k =1 or 3, $e_{2jk} = 0$
  - This leaves j=1, k=3 and j=3, k=1
  - Both of these terms will result in zero because
    - j=1,k=3: $\varepsilon_{3m} = 0$
    - j=3,k=1: $dx_3 = 0$
  - True for both vectors



$$\varepsilon = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega = (0,0,0)$$

$$(1)dx = (1,0,0) = d\hat{x}_1$$

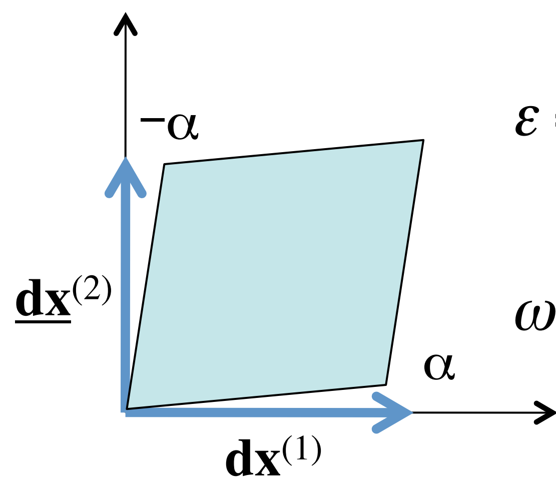
$$(2)dx = (0,1,0) = d\hat{x}_2$$

A Worked Example

- Now for i = 3

$$\Theta_3 = e_{3jk}\left(\varepsilon_{km} + \omega_{km}\right) d\hat{x}_j d\hat{x}_m$$

- Rules for e_{3jk}
  - Only j=1, k=2 and j=2, k=1 are non-zero



$$\varepsilon = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega = (0,0,0)$$

- Vector 1:

$(j = 1, k = 2)$ $\quad e_{312} dx_1 \varepsilon_{2m} dx_m$

$$= 1 \cdot 1 \cdot \left(\alpha \cdot 1 + 0 \cdot 0 + 0 \cdot 0\right) = \alpha$$

$(j = 2, k = 1)$ $\quad e_{321} dx_2 \varepsilon_{1m} dx_m$

$$= -1 \cdot 0 \cdot \left(0 \cdot 1 + \alpha \cdot 0 + 0 \cdot 0\right) = 0$$

- Vector 2:

$(j = 1, k = 2)$ $\quad e_{312} dx_1 \varepsilon_{2m} dx_m$

$$= 1 \cdot 0 \cdot \left(\alpha \cdot 0 + 0 \cdot 0 + 0 \cdot 0\right) = 0$$

$(j = 2, k = 1)$ $\quad e_{321} dx_2 \varepsilon_{1m} dx_m$

$$= -1 \cdot 1 \cdot \left(0 \cdot 0 + \alpha \cdot 1 + 0 \cdot 0\right) = -\alpha$$

$$(1) dx = (1,0,0) = d\hat{x}_1$$

$$(2) dx = (0,1,0) = d\hat{x}_2$$

Rotation of a Line Segment

- There is a general expression for the rotation of a line segment. I'll outline how it is derived without going into all of the details.

$$\Theta_i = e_{ijk}\left(\varepsilon_{km} + \omega_{km}\right) d\hat{x}_j d\hat{x}_m$$

- First, the strain part

$$\Theta_i^{(strain)} = e_{ijk}\varepsilon_{km} d\hat{x}_j d\hat{x}_m = e_{ijk} d\hat{x}_j \left(\varepsilon_{km} d\hat{x}_m\right)$$

$$\Theta_i^{(strain)} = e_{ijk} d\hat{x}_j \left(\varepsilon \cdot d\hat{x}\right)_k$$

$$\Theta^{(strain)} = d\hat{x} \times \left(\varepsilon \cdot d\hat{x}\right)$$

If the strain changes the orientation of the line, then there is a rotation.

Rotation of a Line Segment

- Now the rotation part

$$\Theta_i^{(rot)} = e_{ijk}\omega_{km}d\hat{x}_j d\hat{x}_m = -e_{ijk}e_{kms}\Omega_s d\hat{x}_j d\hat{x}_m$$

$$\Theta_i^{(rot)} = -\left(\delta_{im}\delta_{js} - \delta_{is}\delta_{jm}\right)\Omega_s d\hat{x}_j d\hat{x}_m$$

$$\Theta_i^{(rot)} = -\left(\Omega_j d\hat{x}_j d\hat{x}_i - \Omega_i d\hat{x}_j d\hat{x}_j\right)$$

$$\Theta_i^{(rot)} = \Omega_i - \left(\Omega_j d\hat{x}_j\right)d\hat{x}_i$$

$$\Theta^{(rot)} = \Omega - \left(\Omega \cdot d\hat{x}\right)d\hat{x}$$

  - The reason here that there are two terms relates to the orientation of the rotation axis and the line:
    - Rotation axis normal to line, $\Theta = \Omega$
    - Rotation axis parallel to line, $\Theta = 0$

Rotation of a Line Segment

- Here is the full equation then:

$$\Theta_i = e_{ijk} d\hat{x}_j \varepsilon_{km} d\hat{x}_m + \Omega_i - \left(\Omega_j d\hat{x}_j\right) d\hat{x}_i$$

$$\Theta = d\hat{x} \times \left(\varepsilon \cdot d\hat{x}\right) + \Omega - \left(\Omega \cdot d\hat{x}\right) d\hat{x}$$

- This is used for cases when you have angle or orientation change data, or when you want to predict orientation changes from a known strain and rotation.

Vertical Axis Rotation

- Special case of common use in tectonics
  - We'll use a local east-north-up coordinate system
  - All motion is horizontal ($u_3 = 0$)
  - All rotation is about a vertical axis (only Ω_3 is non-zero)
  - All sites are in horizontal plane ($dx_3 = 0$)
  - The expression for Ω gets a lot simpler

$$\Omega_i - \Omega_j dx_j dx_i \quad \Rightarrow \Omega_i \quad (\Omega \cdot dx = 0)$$

$$\Omega_3 = -\tfrac{1}{2} e_{3ij}\omega_{ij}$$

$$\Omega_3 = -\tfrac{1}{2}\left[e_{312}\omega_{12} + e_{321}\omega_{21} + 0\right]$$

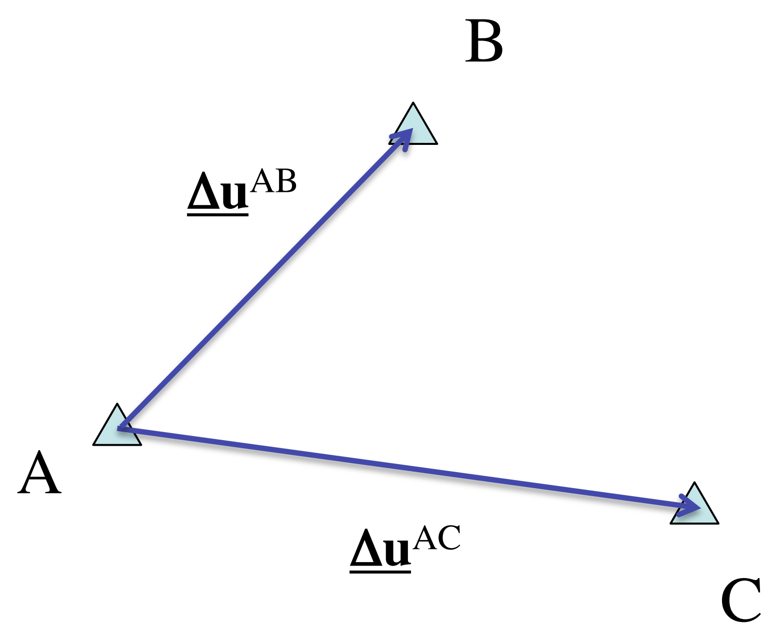
$$\Omega_3 = -\tfrac{1}{2}\left[1 \cdot \omega_{12} - 1 \cdot (-\omega_{12})\right]$$

$$\Omega_3 = -\omega_{12}$$

The vertical axis rotation is directly related to the rotation tensor term

Strain from 3 GPS Sites

- There is a simple, general way to calculate average strain+rotation from 3 GPS sites
- If you have more than 3 sites, divide the network into triangles
  - For example, Delaunay triangulation as implemented in GMT

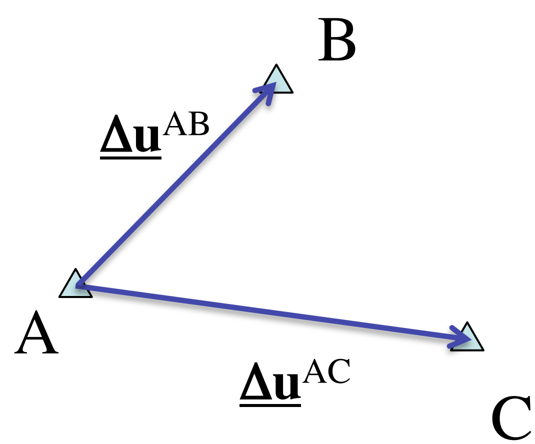


Strain from 3 GPS Sites

- We have seen the equations for strain from a single baseline before

$$\begin{bmatrix}\Delta u_1^{AB}\\ \Delta u_2^{AB}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}\Delta x_1^{AB}+\varepsilon_{12}\Delta x_2^{AB}+\omega_{12}\Delta x_2^{AB}\\ \varepsilon_{12}\Delta x_1^{AB}+\varepsilon_{22}\Delta x_2^{AB}-\omega_{12}\Delta x_1^{AB}\end{bmatrix}$$

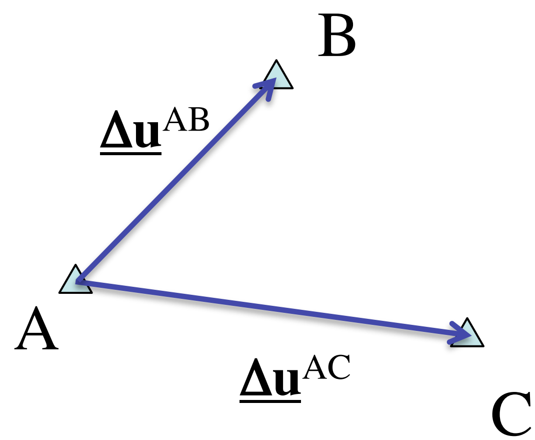
$$\begin{bmatrix}\Delta u_1^{AB}\\ \Delta u_2^{AB}\end{bmatrix}=\begin{bmatrix}\Delta x_1^{AB} & \Delta x_2^{AB} & 0 & \Delta x_2^{AB}\\ 0 & \Delta x_1^{AB} & \Delta x_2^{AB} & -\Delta x_1^{AB}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{12}\\ \varepsilon_{22}\\ \omega_{12}\end{bmatrix}$$



Observations = (known Model)(unknowns)*

Strain from 3 GPS Sites

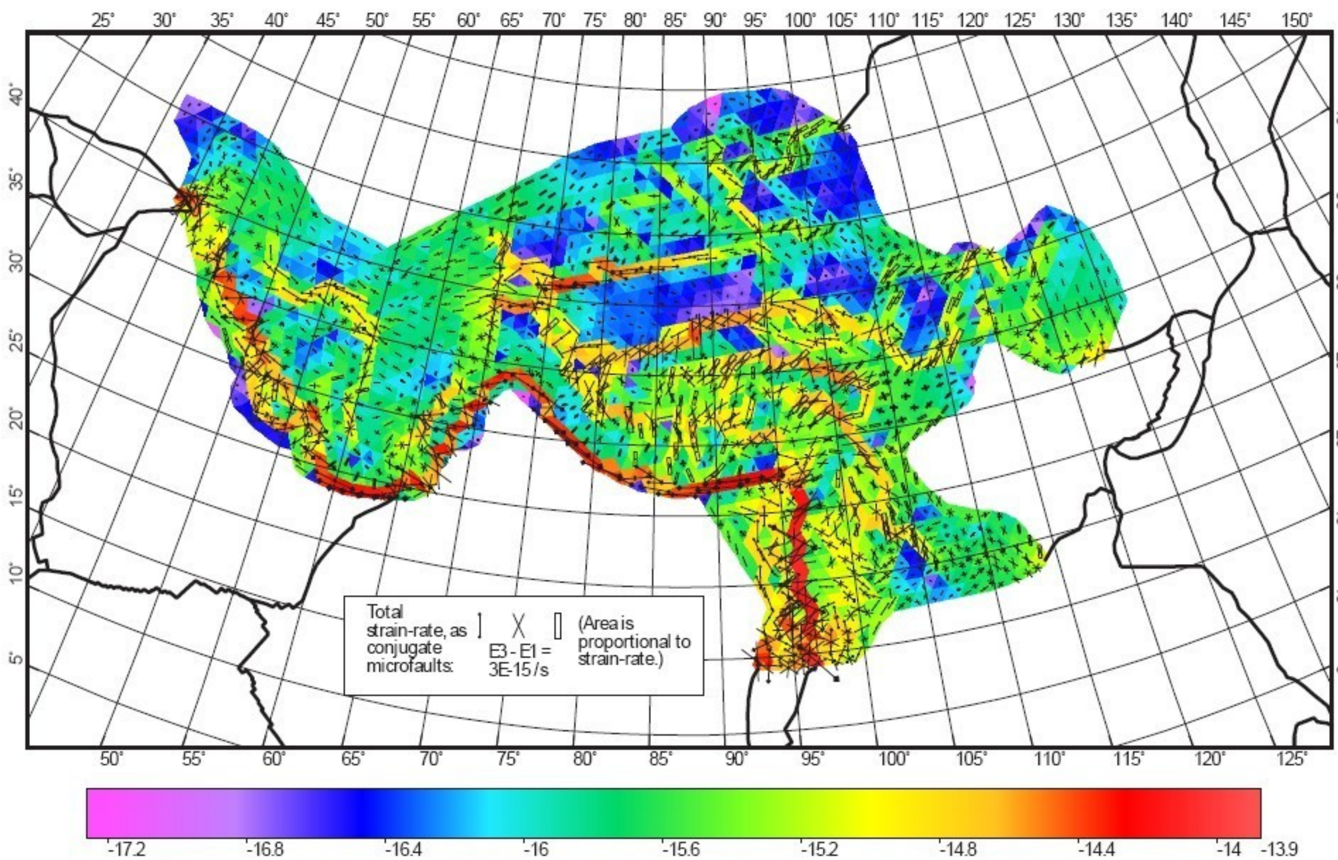
- Including both sites we have 4 equations in 4 unknowns:



$$\begin{bmatrix} \Delta u_1^{AB} \\ \Delta u_2^{AB} \\ \Delta u_1^{BC} \\ \Delta u_2^{BC} \end{bmatrix} = \begin{bmatrix} \Delta x_1^{AB} & \Delta x_2^{AB} & 0 & \Delta x_2^{AB} \\ 0 & \Delta x_1^{AB} & \Delta x_2^{AB} & -\Delta x_1^{AB} \\ \Delta x_1^{BC} & \Delta x_2^{BC} & 0 & \Delta x_2^{BC} \\ 0 & \Delta x_1^{BC} & \Delta x_2^{BC} & -\Delta x_1^{BC} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \omega_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \omega_{12} \end{bmatrix} = \begin{bmatrix} \Delta x_1^{AB} & \Delta x_2^{AB} & 0 & \Delta x_2^{AB} \\ 0 & \Delta x_1^{AB} & \Delta x_2^{AB} & -\Delta x_1^{AB} \\ \Delta x_1^{BC} & \Delta x_2^{BC} & 0 & \Delta x_2^{BC} \\ 0 & \Delta x_1^{BC} & \Delta x_2^{BC} & -\Delta x_1^{BC} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \Delta u_1^{AB} \\ \Delta u_2^{AB} \\ \Delta u_1^{BC} \\ \Delta u_2^{BC} \end{bmatrix}$$

Strain Varies in Space



To increase legibility, tensor icons are plotted for only 1/2 of the finite elements.

Liu, Z., & P. Bird [2008]; doi: 10.1111/j.1365-246X.2007.03640.x

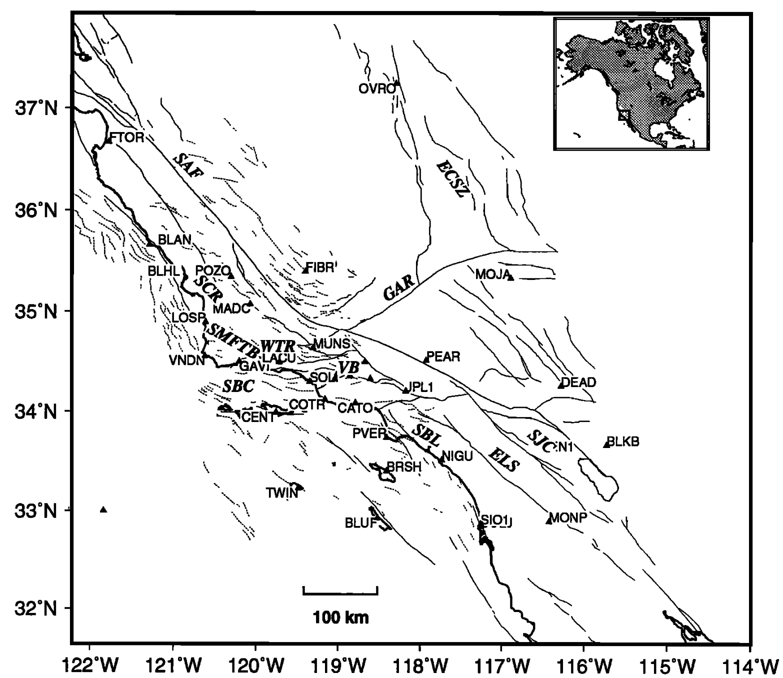
Strain varies with space

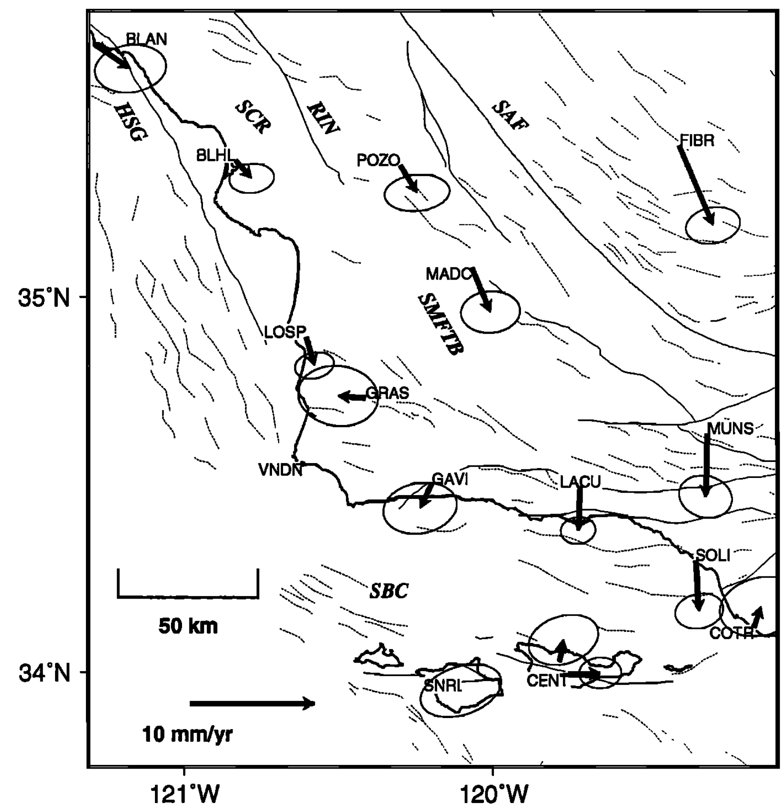
- In general, strain varies with space, so you can't necessarily assume uniform strain over a large area
  - For a strike slip fault, the strain varies with distance from the fault x (slip rate V, and locking depth D)

$$\dot{\varepsilon}_{12} = \frac{VD}{2\pi}\left(x^2 + D^2\right)^{-1}$$

  - You can ignore variations and use a uniform approximation (see no evil)
  - Fit a mathematical model for $\underline{\boldsymbol{\varepsilon}}(\underline{\mathbf{x}})$ and $\underline{\boldsymbol{\omega}}(\underline{\mathbf{x}})$
  - You can try sub-regions, as we did in Tibet in the Chen et al. (2004) paper
  - Map strain by looking at each triangle
  - Or you can map out variations in strain in a more continuous fashion
    - This is a more powerful tool in general

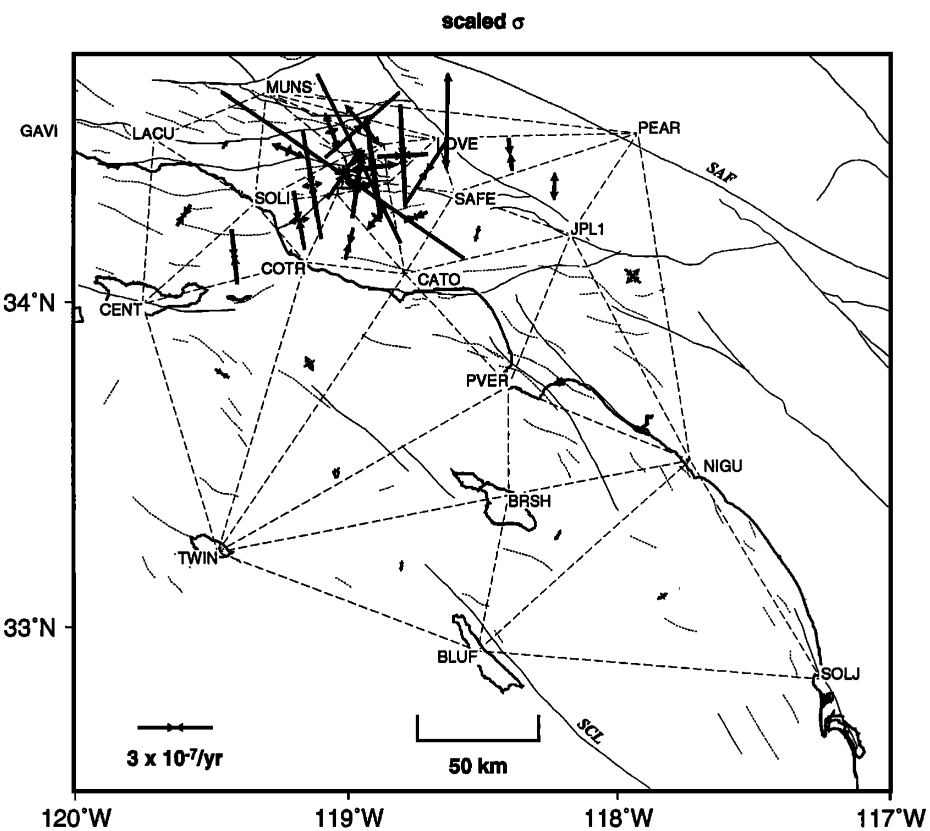
Southern California

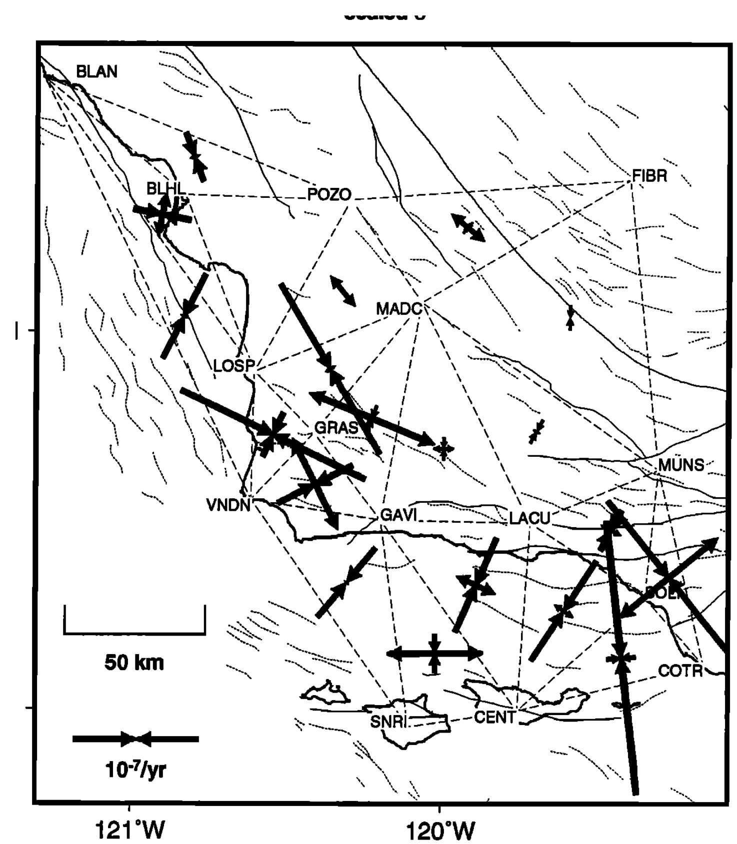




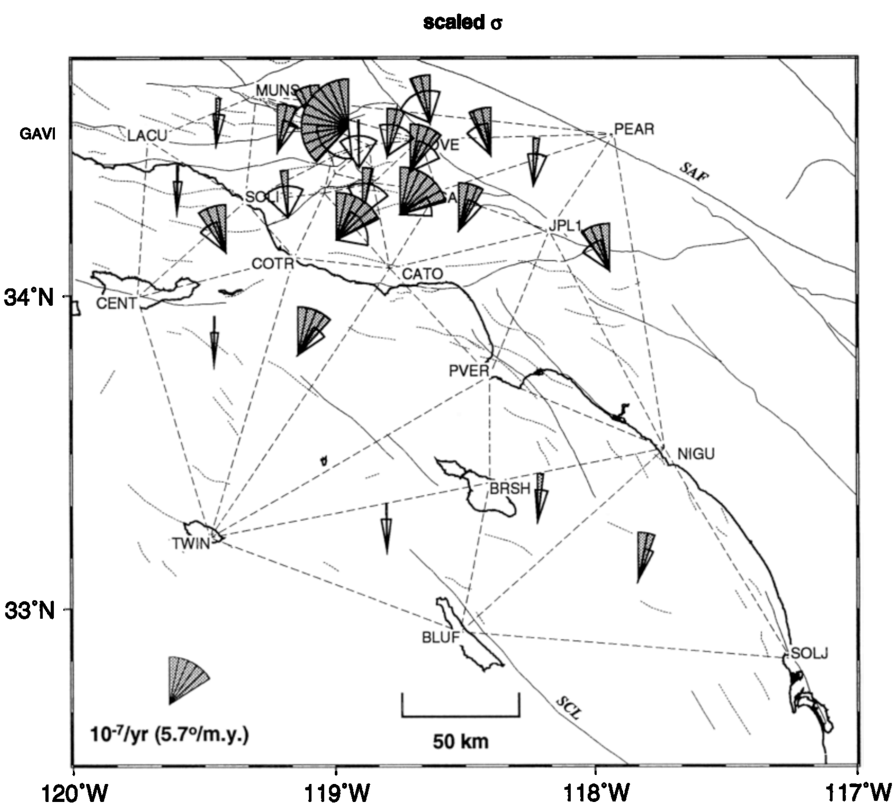
Feigl et al. (1993, JGR)

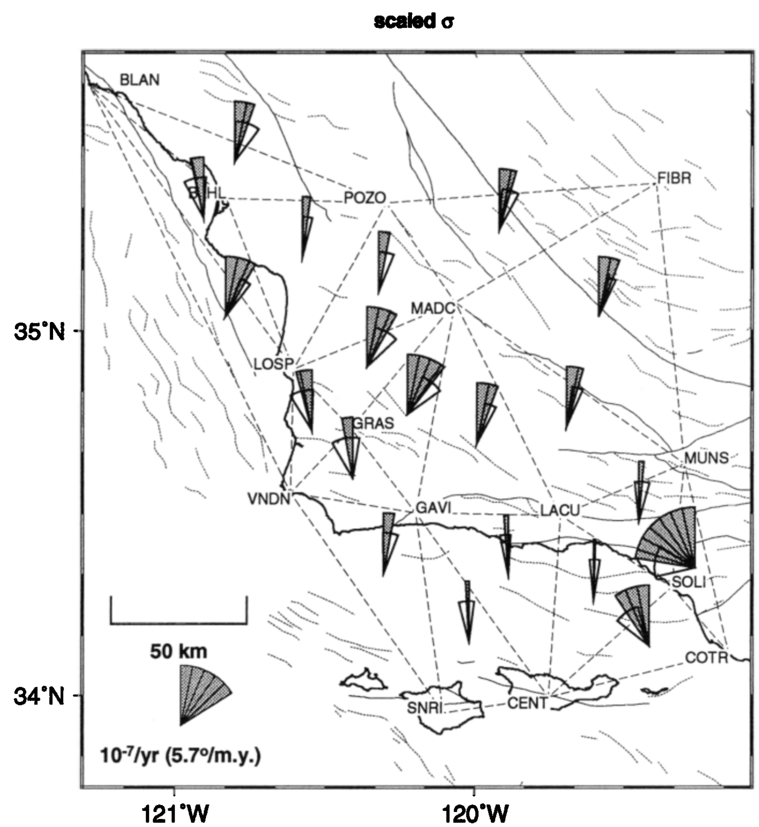
Strain Rates





Rotations


scaled σ
GAVI
LACU
MUNS
PEAR
SAF
JPL1
COTR
CATO
CENT
34°N
PVER
NIGU
BRSH
TWIN
33°N
BLUF
SOLJ
SCL
10^{-7}/yr (5.7°/m.y.)
50 km
120°W
119°W
118°W
117°W


scaled σ
BLAN
POZO
FIBR
MADC
35°N
LOSP
GRAS
MUNS
VNDN
GAVI
LACU
SOLI
50 km
COTR
34°N
SNRI
CENT
10^{-7}/yr (5.7°/m.y.)
121°W
120°W

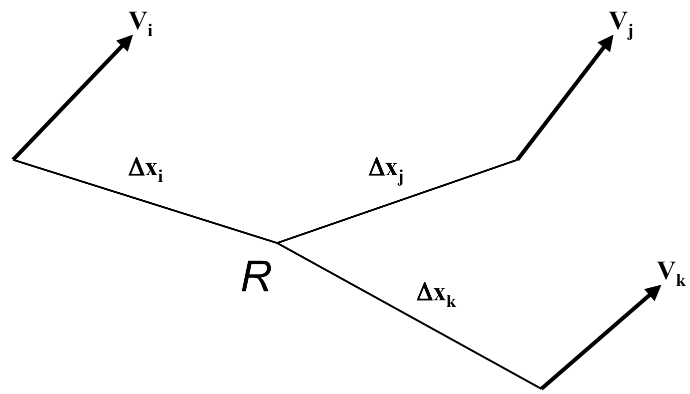
Model strain rates as continuous functions using a modified least-squares method.

Uniqueness of the method:

- Requires no assumptions of stationary of deformation field and uniform variance of the data that many other methods do;
- Implements the degree of smoothing based on *in situ* data strength.
- Method developed by Z. Shen, UCLA

At each location point $\underline{R}$, assuming a uniform strain rate field, the strain rates and the geodetic data can be linked by a linear relationship:

$$\underline{d} = A\,\underline{m} + \underline{e}$$



$$\begin{bmatrix} Vx_1 \\ Vy_1 \\ Vx_2 \\ Vy_2 \\ \cdots \\ Vx_n \\ Vy_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x_1 & \Delta y_1 & 0 & \Delta y_1 \\ 0 & 1 & 0 & \Delta x_1 & \Delta y_1 & -\Delta x_1 \\ 1 & 0 & \Delta x_2 & \Delta y_2 & 0 & \Delta y_2 \\ 0 & 1 & 0 & \Delta x_2 & \Delta y_2 & -\Delta x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \Delta x_n & \Delta y_n & 0 & \Delta y_n \\ 0 & 1 & 0 & \Delta x_n & \Delta y_n & -\Delta x_n \end{bmatrix} \begin{bmatrix} Ux \\ Uy \\ \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{yy} \\ \omega \end{bmatrix} + \begin{bmatrix} e_{x1} \\ e_{y1} \\ e_{x2} \\ e_{y2} \\ \cdots \\ e_{xn} \\ e_{yn} \end{bmatrix}$$

Ux, Uy: on spot velocity components

$\tau\,xx$, τxy, τyy: strain rate components

ω: rotation rate

$$\underline{d} = A\,\underline{m} + \underline{e}$$

reconstitute the inverse problem with a weighting matrix *B*:

$$B\underline{d} = BA\underline{m} + B\underline{e}$$

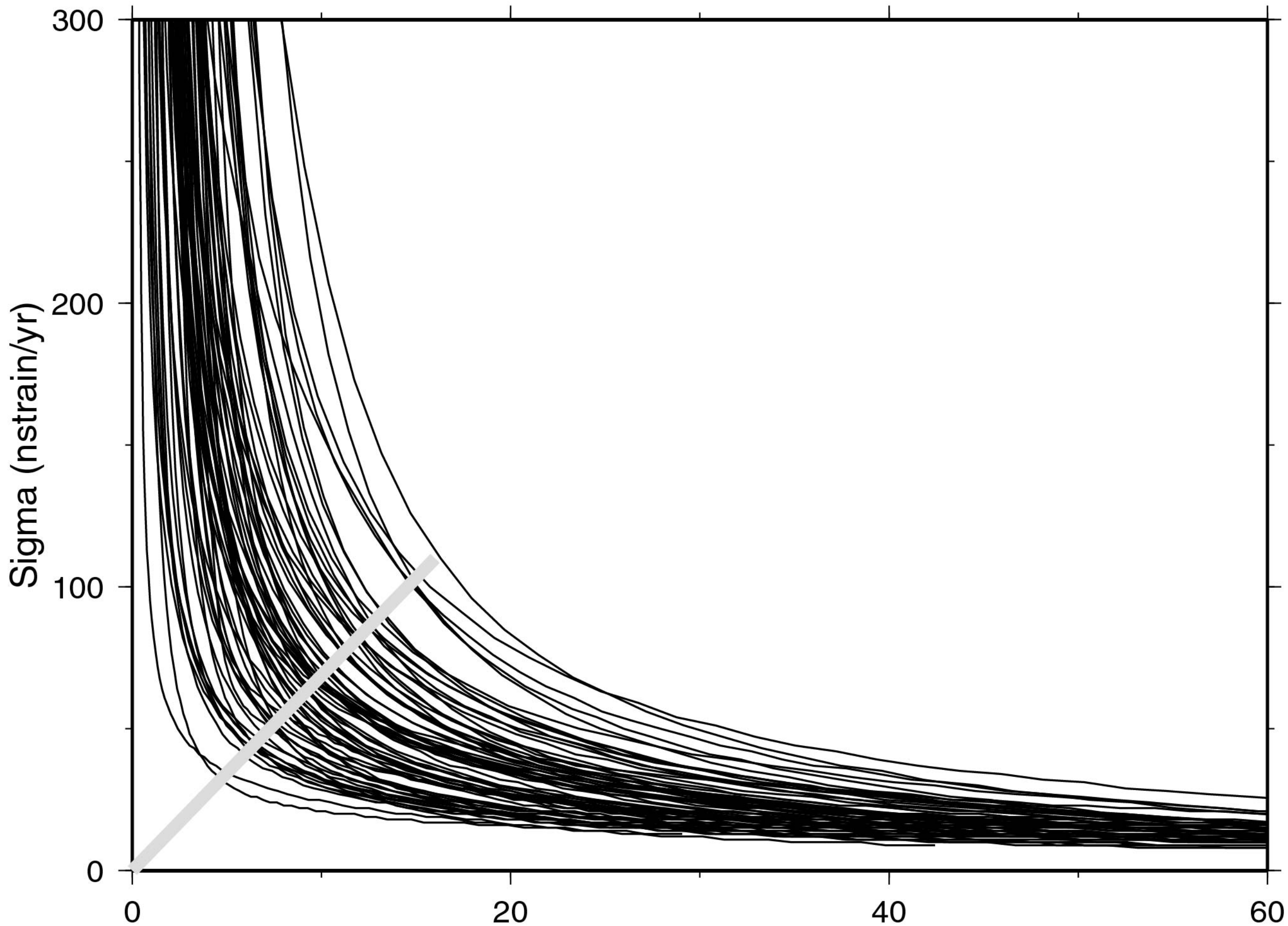
where *B* is a diagonal matrix whose *i*-th diagonal term is $exp(-\Delta x_i^2/D^2)$ and $\underline{e} \sim N(0, E)$, *that is, the errors are Assumed* to be normally distributed. *D* is a smoothing distance.

$$\underline{m} = (A^t\, B\, E^{-1}\, B\, A)^{-1}\, A^t\, B\, E^{-1}\, B\, \underline{d}$$

This result comes from standard least squares estimation methods.

The question is how to make a proper assignment of *D*?

Trade-off between total weight W and strain rate uncertainty σ



small D
Average over small area

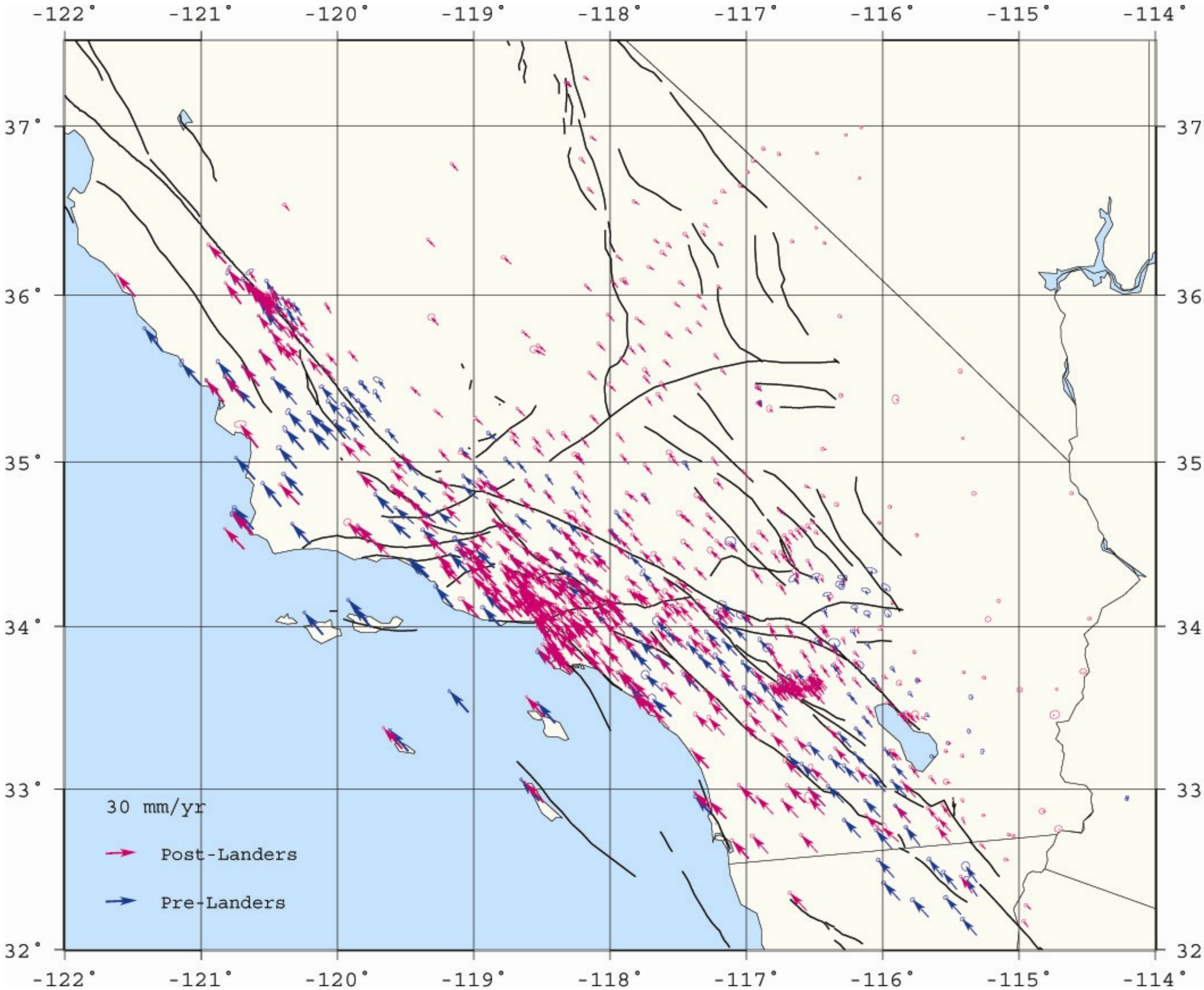
$$W = \Sigma_i \exp(-\Delta x_i^2/D^2)$$

large D
Average over large area

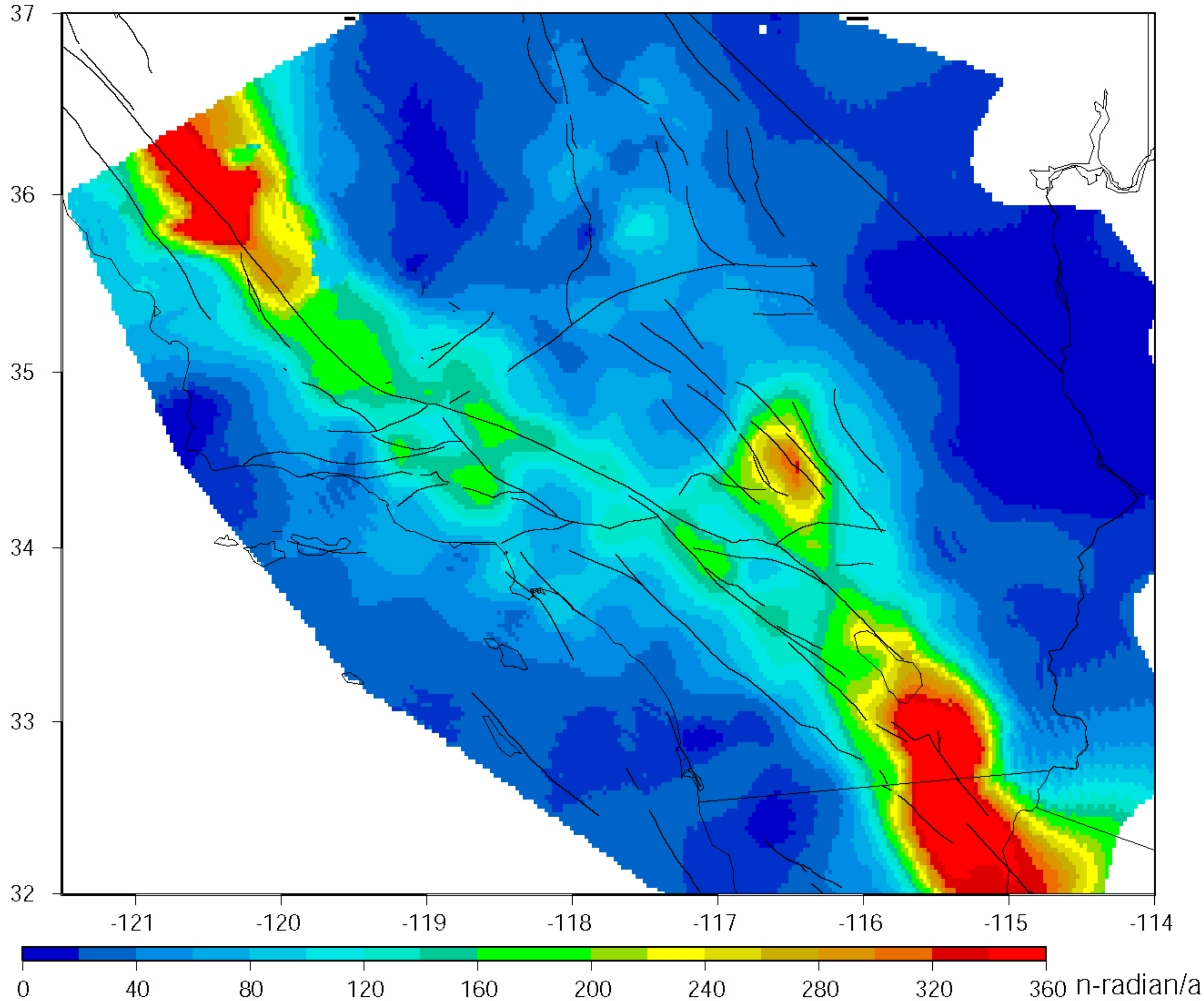
Tradeoff of resolution and uncertainty

- If you average over a small area, you will improve your spatial resolution for variations in strain
  - But your uncertainty in strain estimate will be higher because you use less data **= more noise**
- If you average over a large area, you will reduce the uncertainty in your estimates by using more data
  - But your ability to resolve spatial variations in strain will be reduced **= possible over-smoothing**
- Need to find a balance that reflects the actual variations in strain.

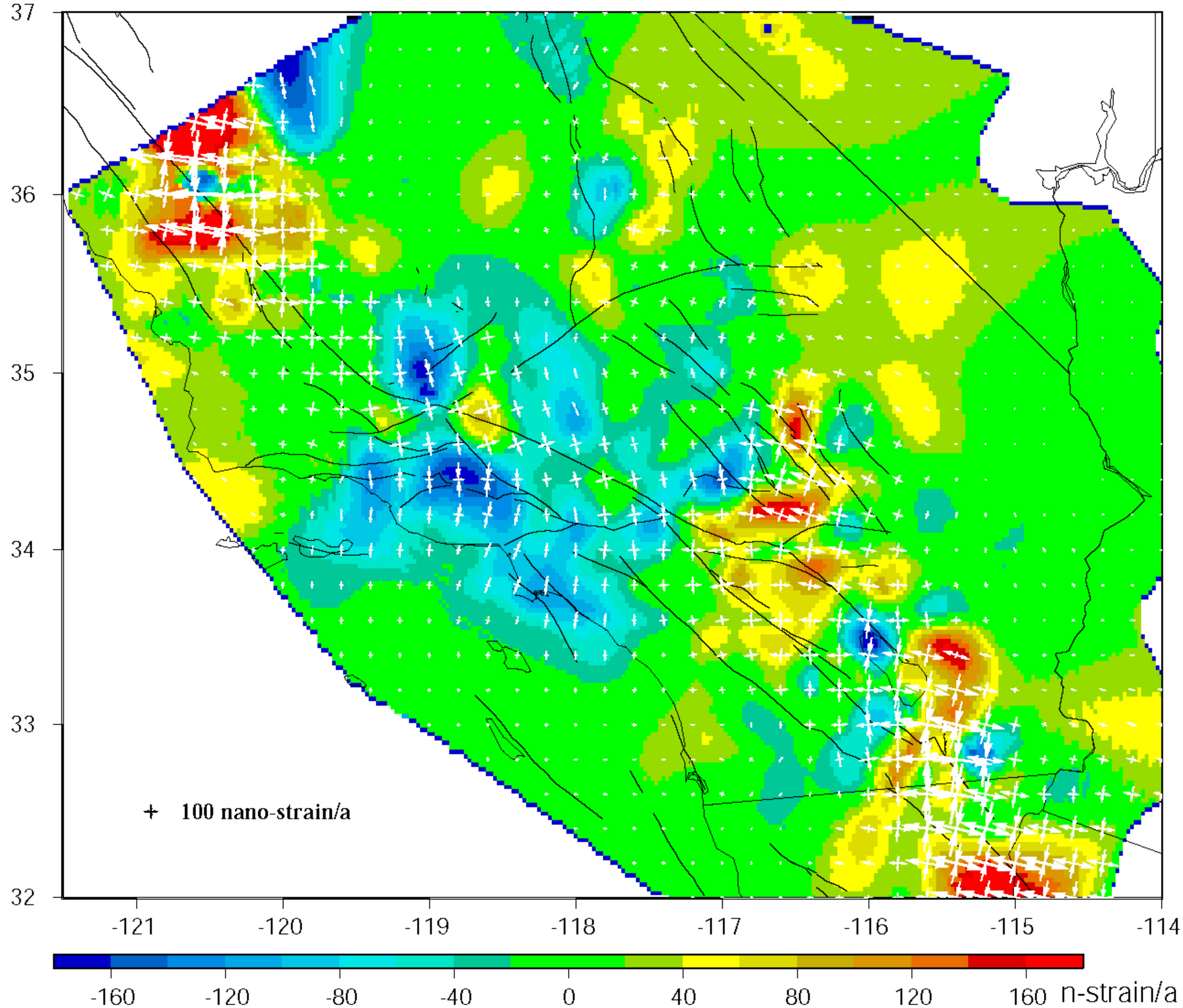
Example: Strain rate estimation from SCEC CMM3 (Post-Landers)



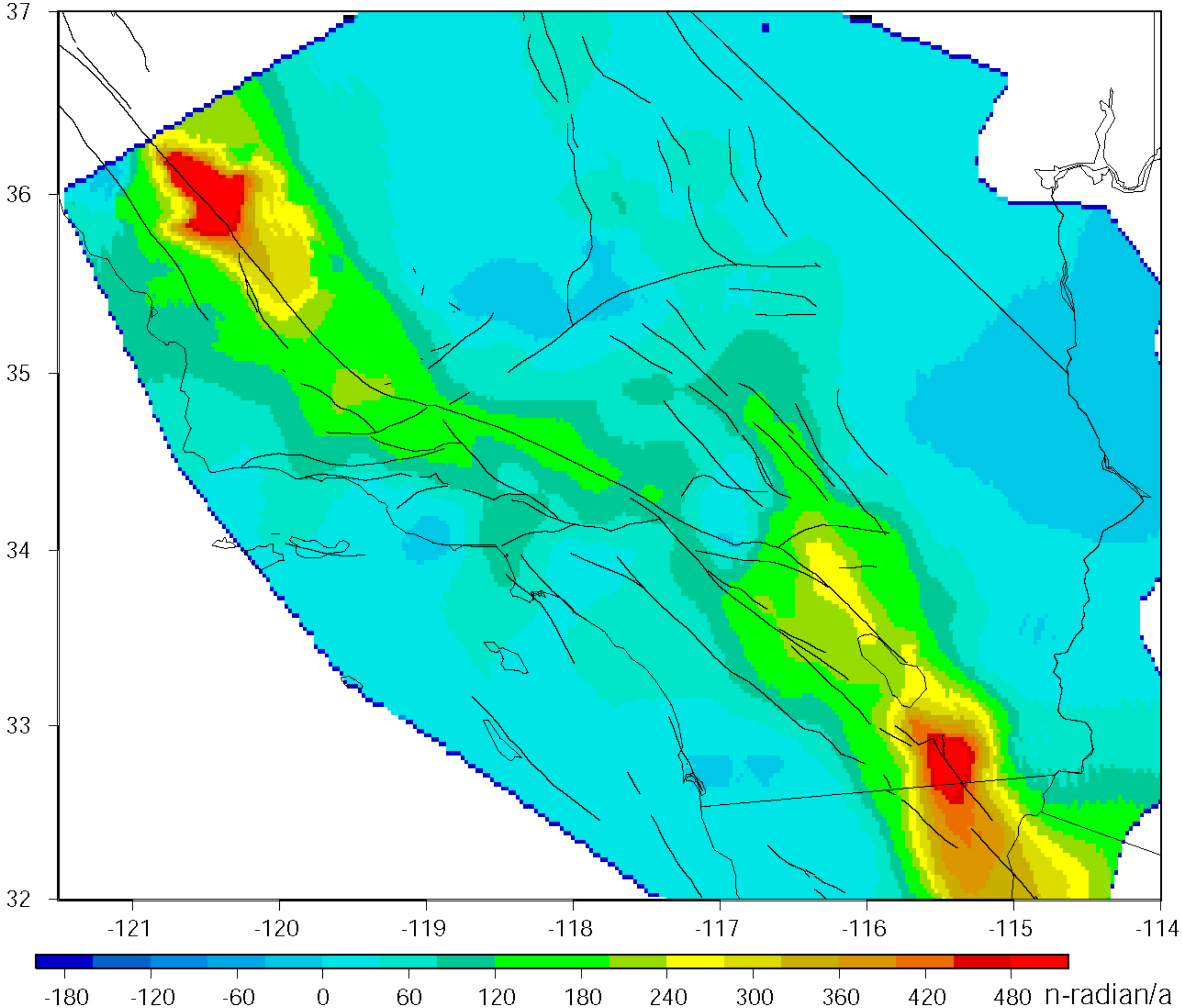
Post-Landers Maximum Shear Strain Rate



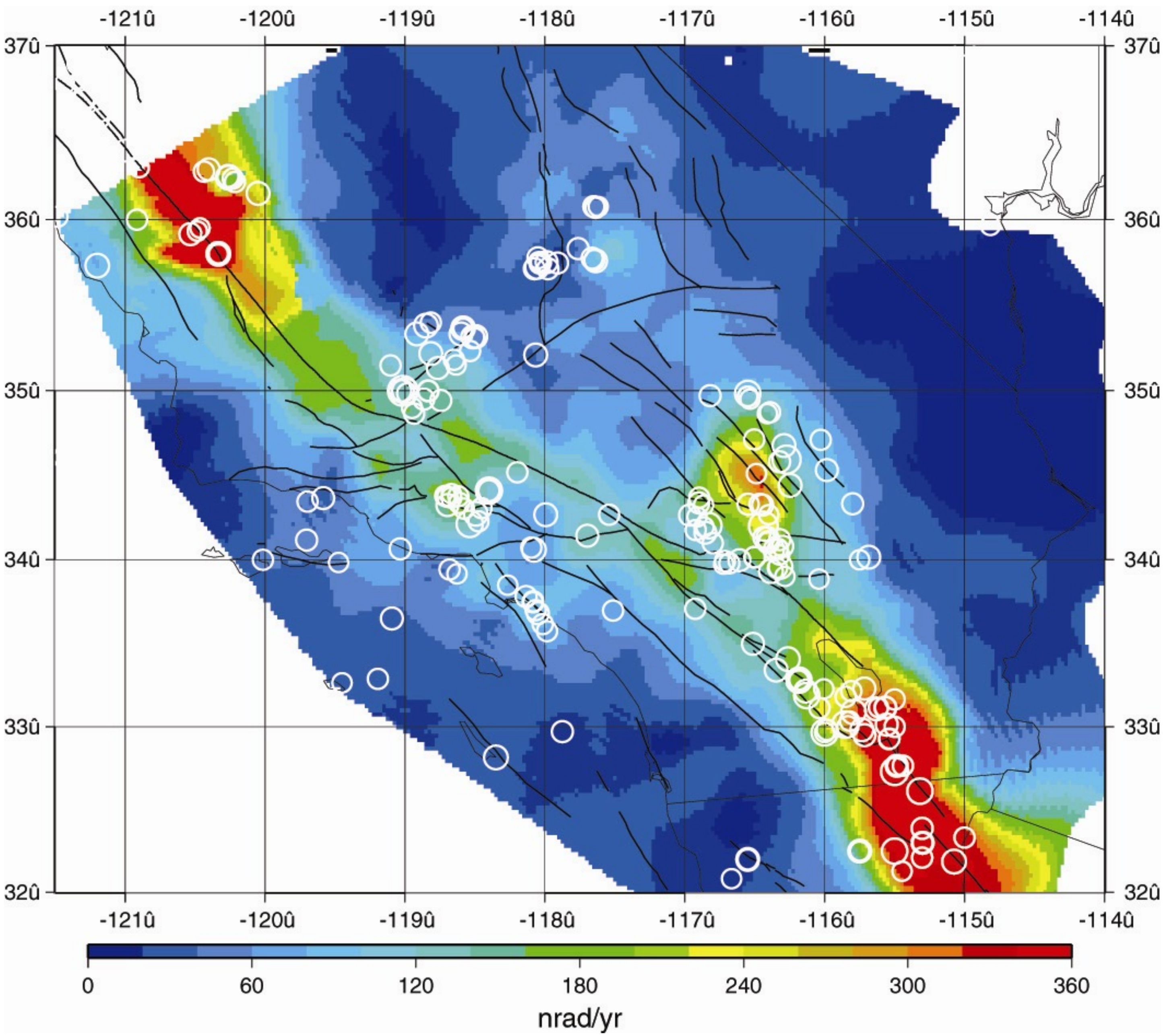
Post-Landers Principal Strain Rate and Dilatation Rate



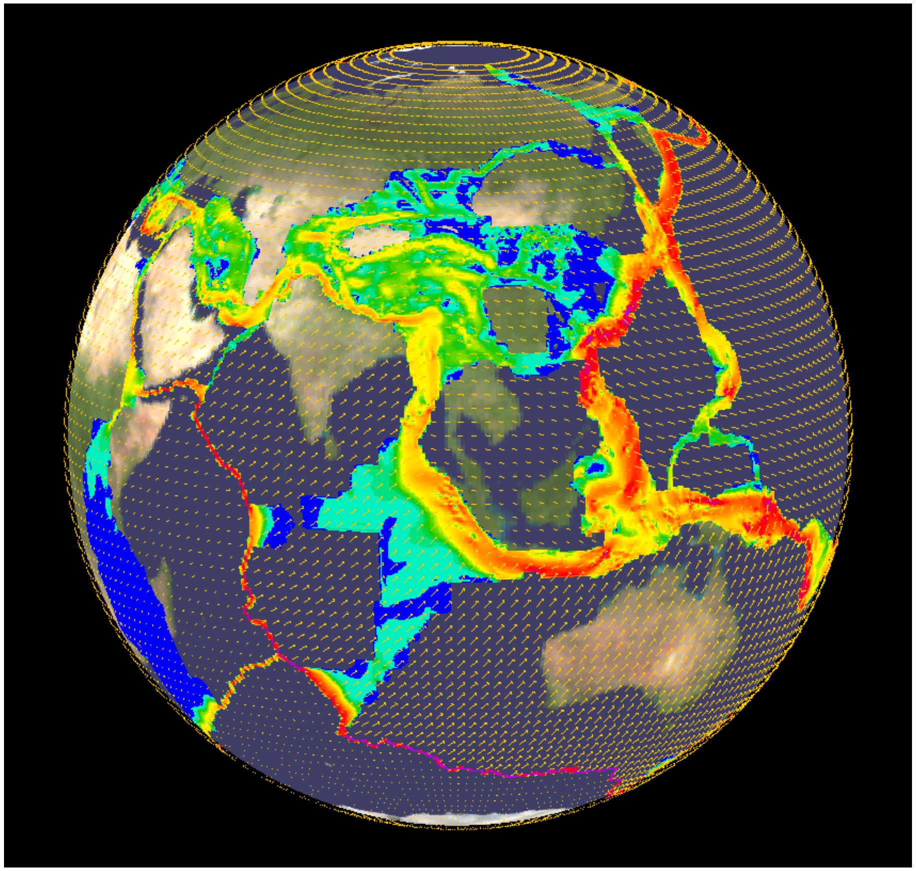
Post-Landers Rotation Rate



Post-Landers Maximum Strain Rate and Earthquakes of M>5.0 1950-2000



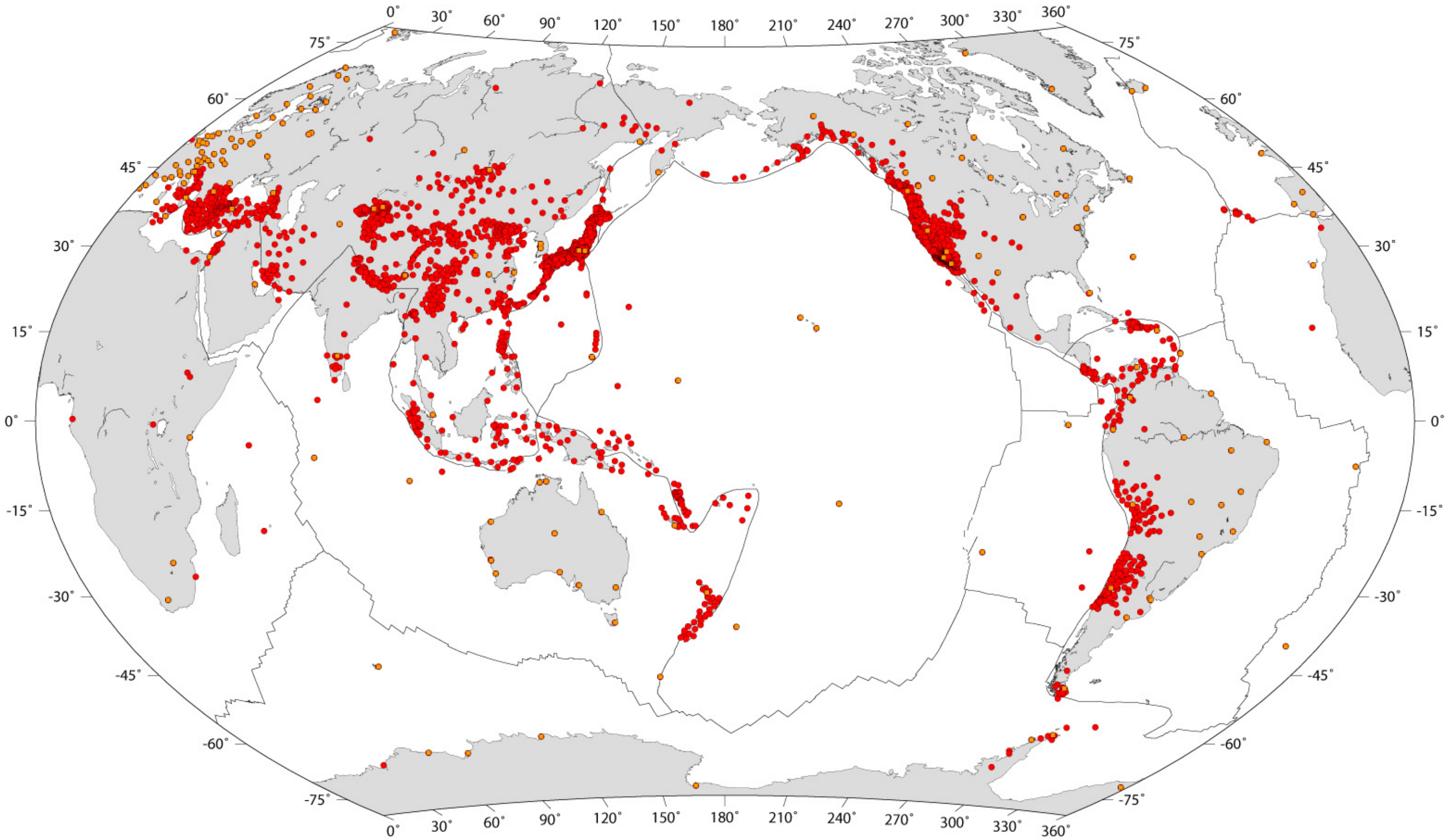
Global Strain Rate Map



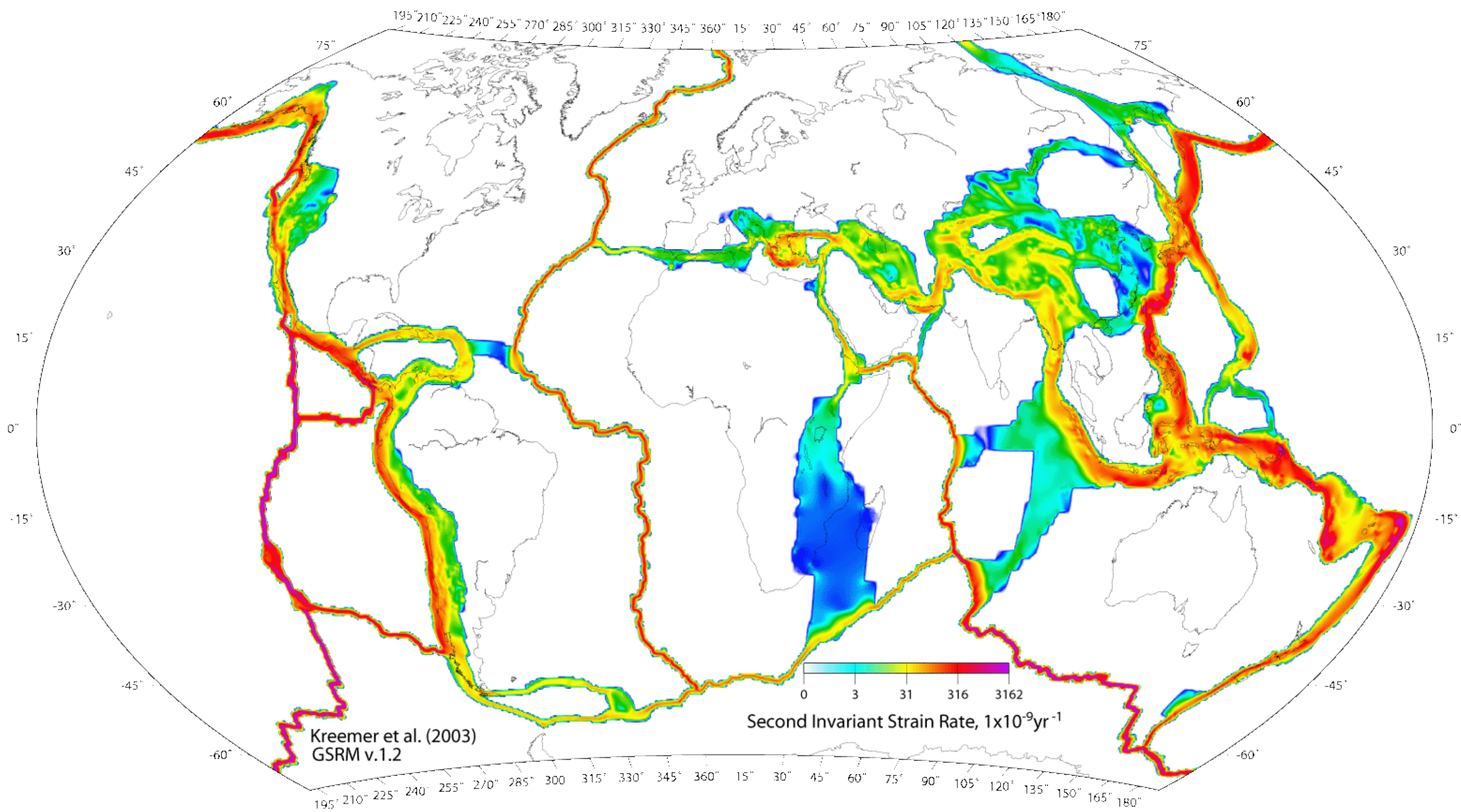
Large-Scale Strain Maps

- The Global Strain Rate Map makes use of some important properties of strain as a function of space:
  - Compatibility equations
    - The strain tensors at two nearby points have to be related to each other if there are no gaps or overlaps in the material
    - Equations relate spatial derivatives of various strain tensor components
  - If we know strain everywhere, we can determine rotation from variations in shear strain (because of compatibility eqs)
  - Relationship between strain and variations in angular velocity, so that you can represent all motion in terms of spatially-variable angular velocity
  - "Kostrov summation" of earthquake moment tensors or fault slip rate estimates
    - Allows the combination of geodetic and geologic/seismic data

Geodetic Velocities Used



Second Invariant of Strain



Invariants of Strain Tensor

- The components of the strain tensor depend on the coordinate system
  - For example, tensor is diagonal when principal axes are used to define coordinates, not diagonal otherwise
- There are combinations of tensor components that are invariant to coordinate rotations
  - Correspond to physical things that do not change when coordinates are changed
  - Dilatation is first invariant (volume change does not depend on orientation of coordinate axes)

Three Invariants (Symmetric)

- First Invariant – dilatation
  - Trace of strain tensor is invariant
    - $I_1 = \Delta = trace(\varepsilon) = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$
- Second Invariant – "Magnitude"

$$I_2 = \tfrac{1}{2}\left[trace(\varepsilon)^2 - trace(\varepsilon \cdot \varepsilon)\right]$$

$$I_2 = \varepsilon_{11} \cdot \varepsilon_{22} + \varepsilon_{22} \cdot \varepsilon_{33} + \varepsilon_{11} \cdot \varepsilon_{33} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{13}^2$$

- Third Invariant – determinant
  - Determinent does not depend on coordinates
    - $I_3 = \det(\varepsilon)$
- There is also a mathematical relationship between the three invariants

Second Invariant of Strain