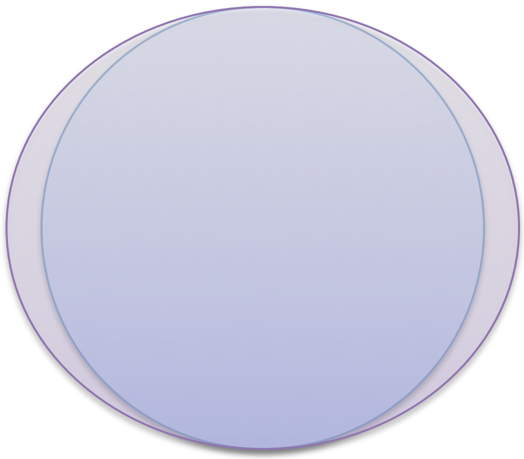
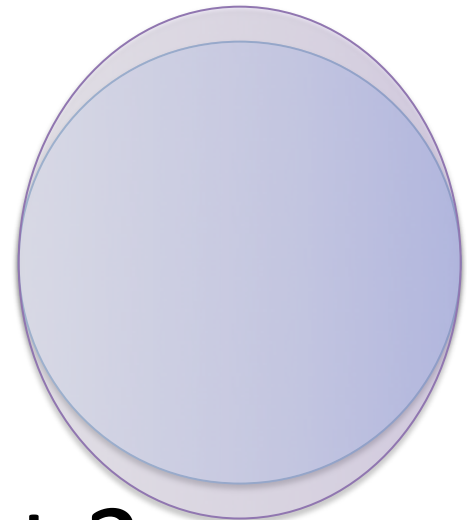
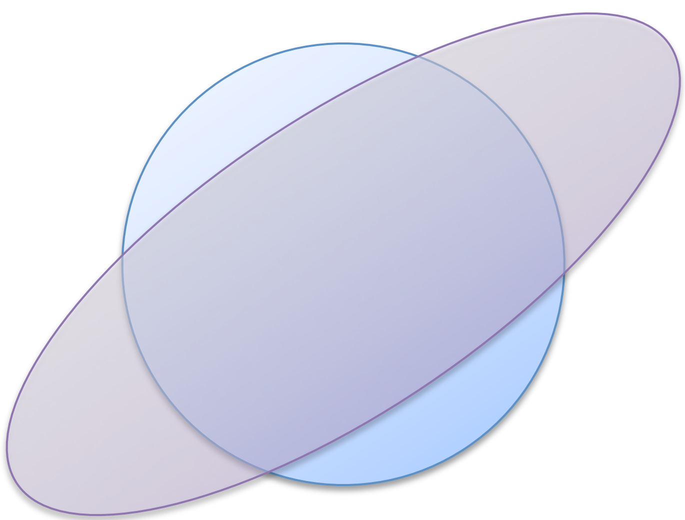




Lecture 13: Strain part 2

GEOS 655 Tectonic Geodesy

Jeff Freymueller



Strain and Rotation Tensors

- We described the deformation as the sum of two tensors, a strain tensor and a rotation tensor.

$$u_i\left(\underline{x_0} + \underline{dx}\right) = u_i\left(\underline{x_0}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)dx_j$$

$$u_i\left(\underline{x_0} + \underline{dx}\right) = u_i\left(\underline{x_0}\right) + \varepsilon_{ij}dx_j + \omega_{ij}dx_j$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

*symmetric, **strain*** *anti-symmetric, **rotation***

OK, So What is a Tensor, Anyway

- Tensor, not tonsure! ➔
- Examples of tensors of various ranks:
  - Rank 0: scalar
  - Rank 1: vector
  - Rank 2: matrix
  - A tensor of rank N+1 is like a set of tensors of rank N, like you can think of a matrix as a set of column vectors
- The stress tensor was actually the first tensor -- the mathematics was developed to deal with stress.
- The mathematical definition is based on transformation properties.

And Writing Them in Index Notation

- We can write the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

- The strain tensor is symmetric, so only 6 components are independent. Symmetry requires that:

$$\varepsilon_{21} = \varepsilon_{12}$$
$$\varepsilon_{31} = \varepsilon_{13}$$
$$\varepsilon_{23} = \varepsilon_{32}$$

- The rotation tensor is anti-symmetric and has only 3 independent components:

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

Estimating Strain and Rotation from GPS Data

- This is pretty easy. We can estimate all of the components of the strain and rotation tensors directly from the GPS data.
  - We can write equations in terms of the 6 independent strain tensor components and 3 independent rotation tensor components
  - Or we can write equations in terms of the 9 components of the displacement gradient tensor
- Write the motions (relative to a reference site or reference location) in terms of distance from reference site:

$$u_i\left(\underline{x_0} + \underline{dx}\right) - u_i\left(\underline{x_0}\right) = \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

- Here $\underline{\mathbf{x_0}}$ is the position of the reference location, and $\underline{\mathbf{dx}}$ is the vector from reference location to where we have data

The Equations in 2D

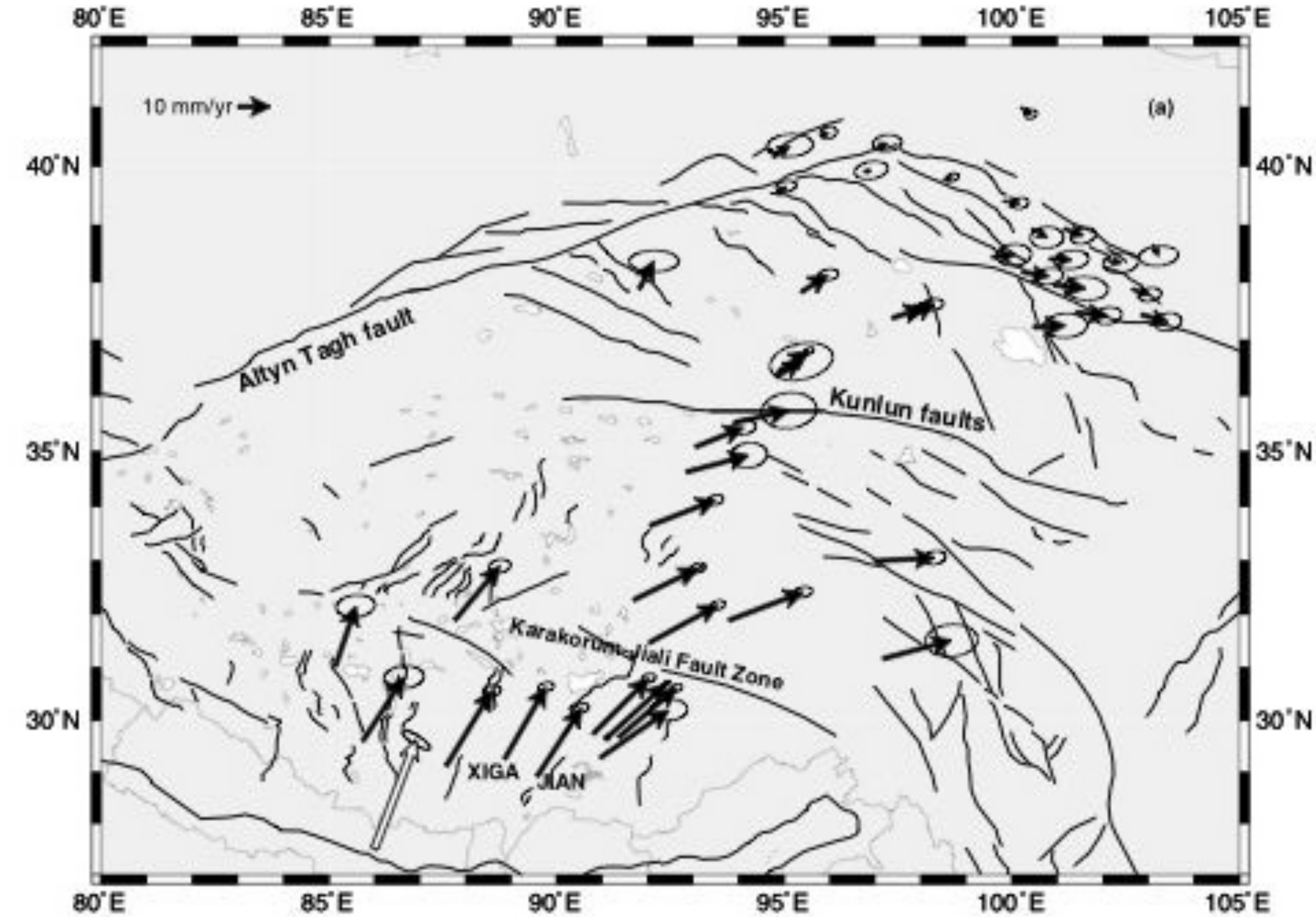
- The equations below are written out for the 2D case (velocity and strain/rotation rate):

$$v_x = V_x + \dot{\varepsilon}_{xx}\Delta x + \dot{\varepsilon}_{xy}\Delta y + \dot{\omega}\Delta y$$

$$v_y = V_y + \dot{\varepsilon}_{xy}\Delta x + \dot{\varepsilon}_{yy}\Delta y - \dot{\omega}\Delta x$$

  - For 2D strain, we have 4 parameters (3 strain rates, 1 rotation rate) and we need ≥2 sites with horizontal data.
  - For 3D, we would have 9 parameters and would require ≥3 sites with 3D data.

Velocities Relative to Eurasia



Chen et al. (2004), JGR

Displacement and Strain

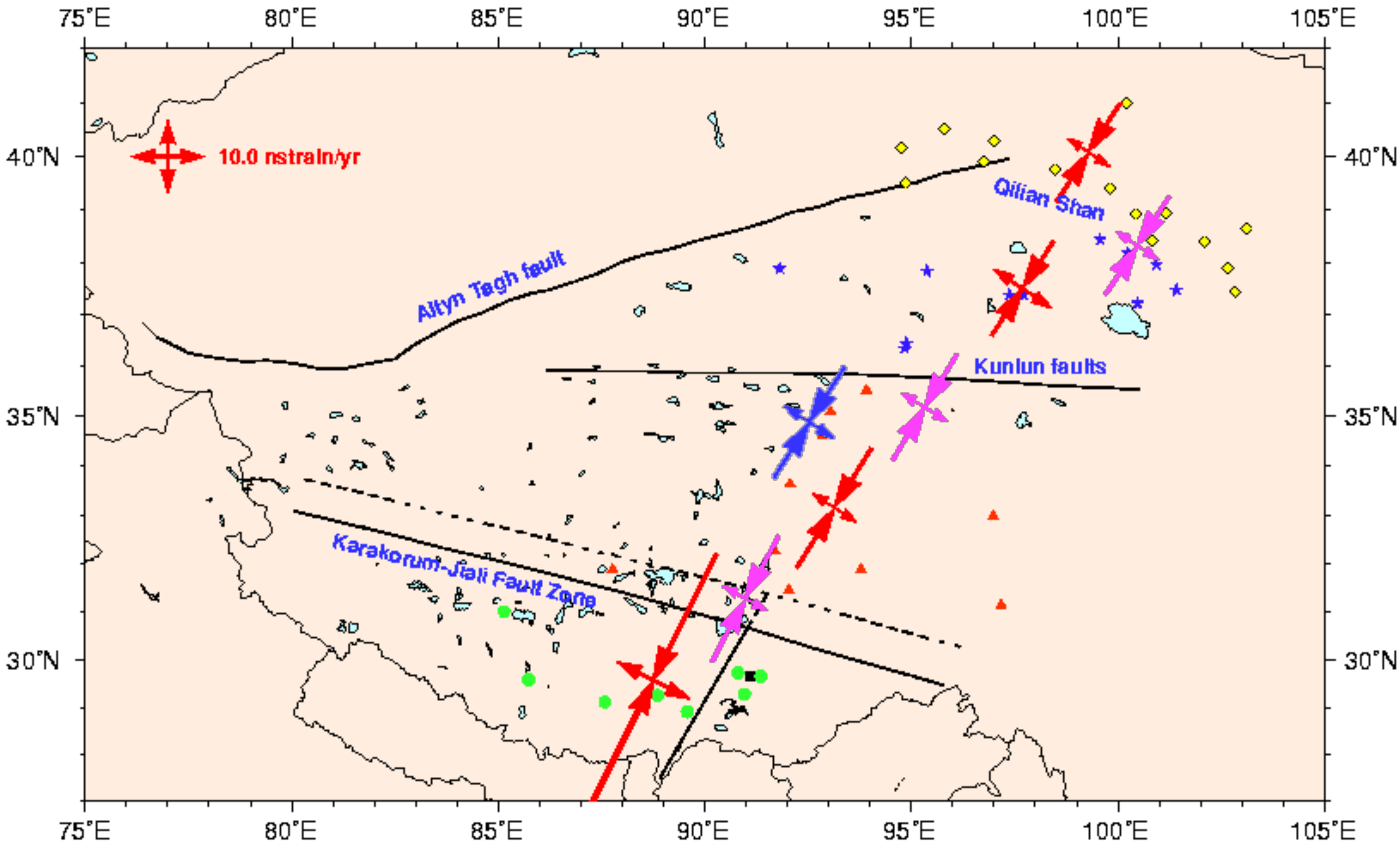
- Displacements (or rates) are a combination of rigid body translation, rotation and internal deformation

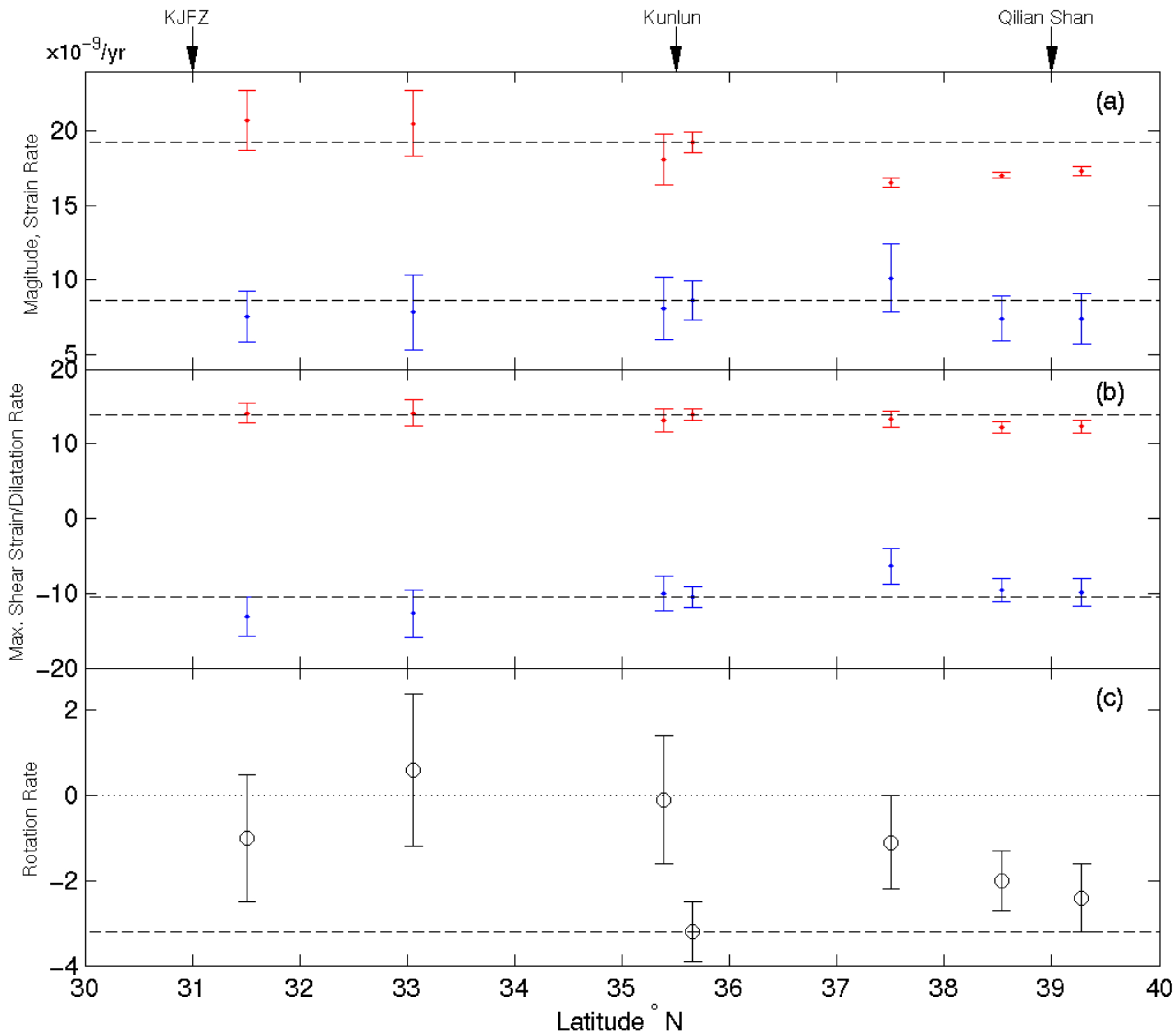
$$\begin{bmatrix} v_{east} \\ v_{north} \end{bmatrix} = \begin{bmatrix} v_{e,body} \\ v_{n,body} \end{bmatrix} + \begin{bmatrix} \dot{\varepsilon}_{11} & \frac{1}{2}\left(\dot{\varepsilon}_{12} + \dot{\omega}\right) \\ \frac{1}{2}\left(\dot{\varepsilon}_{12} - \dot{\omega}\right) & \dot{\varepsilon}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ε = strain tensor components
ω = rotation

(x, y) = position
v = velocity

Uniform Strain Rate

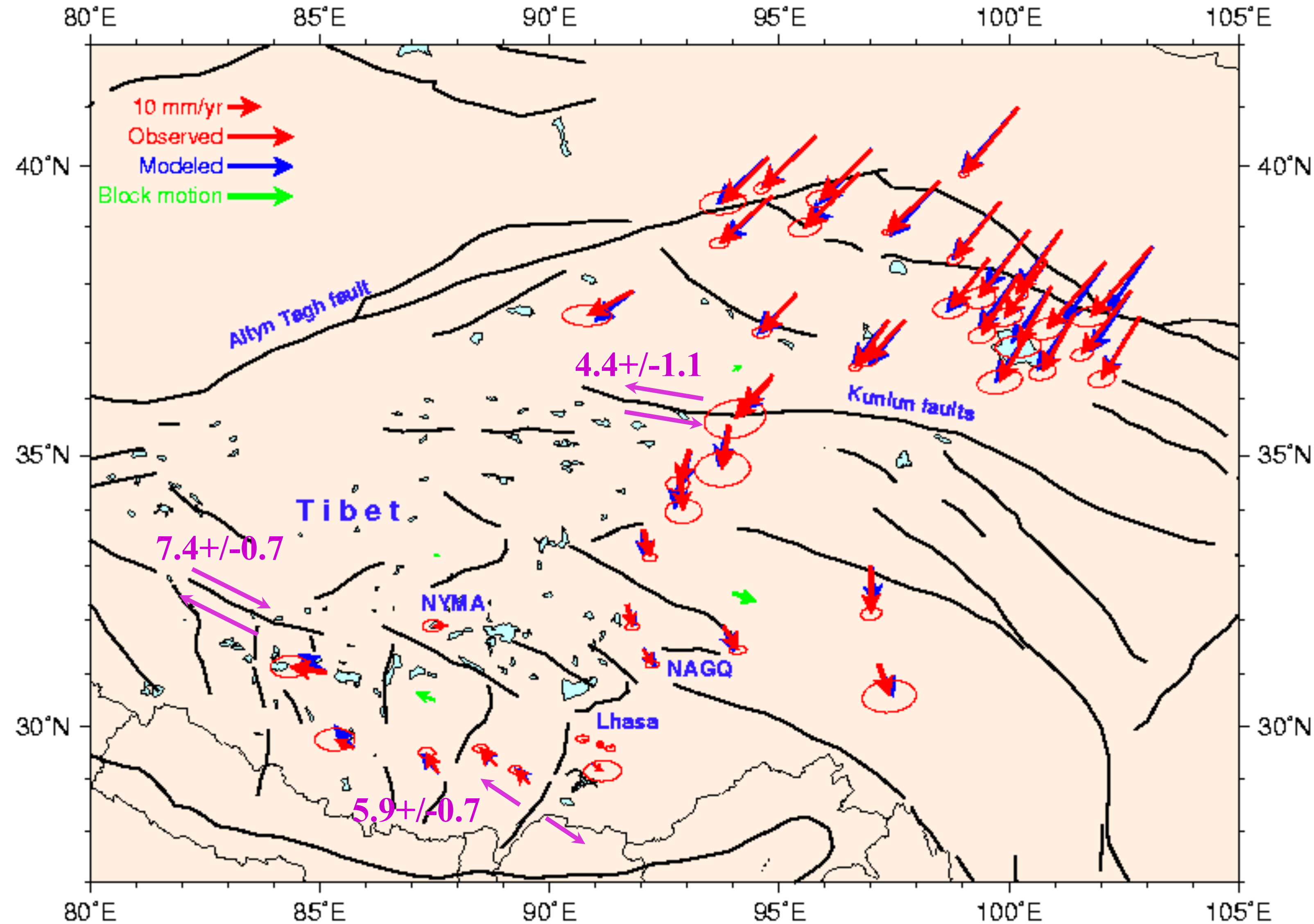




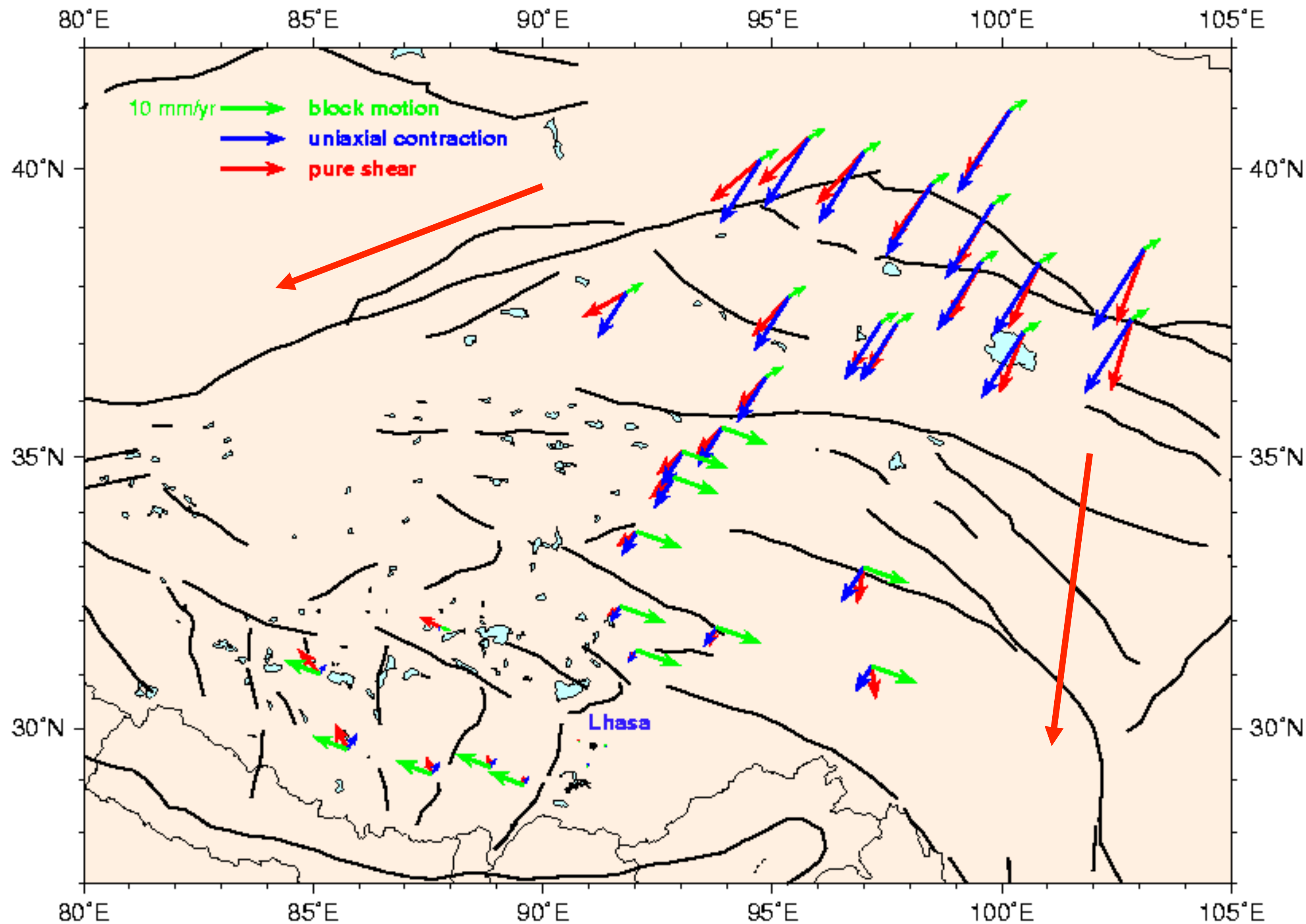
$$\begin{pmatrix} \dot{\varepsilon}_1 & 0 \\ 0 & \dot{\varepsilon}_2 \end{pmatrix} = \dot{\varepsilon}_2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \dot{\varepsilon}_2 \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Deforming Block Model

- Based on GPS data from 44 sites
- Four blocks moving relative to each other on major faults, plus uniform strain
  - Block motions are predominantly strike-slip
- Models with spatial variations in strain do not fit significantly better than uniform strain
- Models with all slip concentrated on a few faults fit worse than the deforming block model.

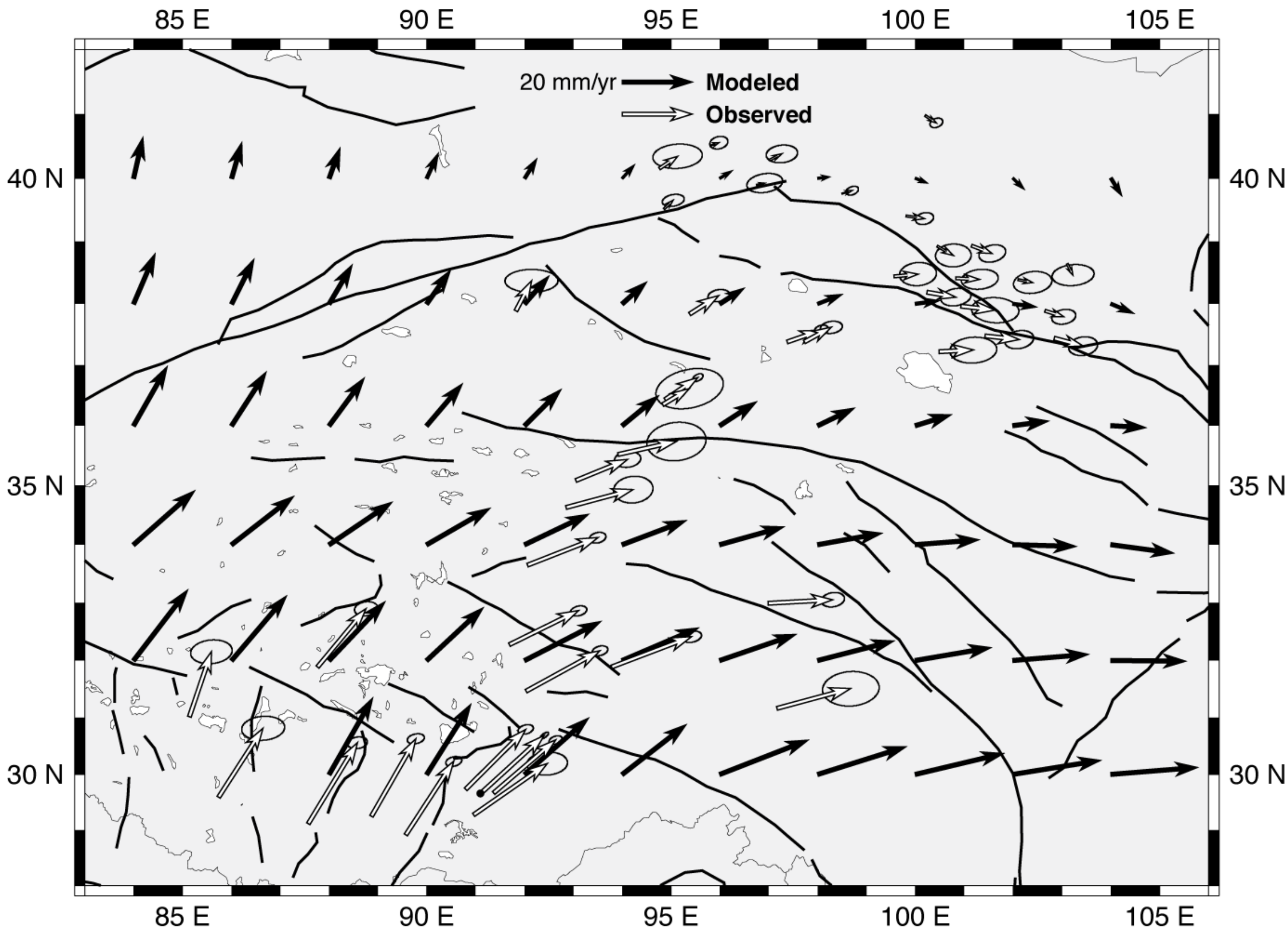


80°E
85°E
90°E
95°E
100°E
105°E
40°N
35°N
30°N
10 mm/yr
Observed
Modeled
Block motion
Altyn Tagh fault
Kunlun faults
Tibet
4.4+/-1.1
7.4+/-0.7
5.9+/-0.7
NYMA
NAGQ
Lhasa



80°E
85°E
90°E
95°E
100°E
105°E
40°N
35°N
30°N
10 mm/yr
block motion
uniaxial contraction
pure shear
Lhasa

Chen et al. Deforming Block Model



Rotation Tensor

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

In terms of our earlier coordinate rotation matrix:

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix} = \left(\beta_{ij}\right)\underline{x} - \underline{x}$$

Rotation as a Vector

- We can represent the rotation tensor as a vector. Why?
  - Rotation can be described by a vector; think angular velocity vector
  - With only three independent components, the number of terms adds up
- We can write a cross product operator using the "permutation tensor" (actually a tensor of rank 3, like a matrix with 3 dimensions): $\Omega_k = -\frac{1}{2} e_{ijk} \omega_{ij}$

$$e_{ijk} = \left\{ \begin{array}{c} e_{123} = e_{231} = e_{312} = 1 \\ e_{213} = e_{321} = e_{132} = -1 \\ otherwise := 0 \end{array} \right\}$$

More on the Permutation Tensor

- We can write the vector cross product in terms of this permutation tensor

$$\left(a \times b\right)_k = e_{ijk} a_i b_j$$

- Remember this rule?

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

  – The permutation tensor is like a short-hand for that rule

- The permutation tensor can be written in terms of the Kronecker delta (this is sometimes useful):

$$e_{ijk} e_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}$$

Ω and ω_{ij}

- We can write Ω in terms of u:

$$\Omega_k = -\tfrac{1}{2} e_{ijk}\omega_{ij} = -\tfrac{1}{2} e_{kij}\omega_{ij}$$

$$\Omega_k = -\tfrac{1}{2}\left(\tfrac{1}{2} e_{kij}\frac{\partial u_i}{\partial x_j} - \tfrac{1}{2} e_{kij}\frac{\partial u_j}{\partial x_i}\right)$$

$$\Omega_k = \tfrac{1}{2}\left(\tfrac{1}{2} e_{kji}\frac{\partial u_i}{\partial x_j} + \tfrac{1}{2} e_{kij}\frac{\partial u_j}{\partial x_i}\right)$$

Notice the permutation

$$\Omega_k = \tfrac{1}{2}\left(e_{kji}\frac{\partial u_i}{\partial x_j}\right)$$

These two terms are equal. Why?

$$\Omega = \tfrac{1}{2}\left(\nabla \times u\right)$$

It is the curl of u

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

ω_{ij} and Ω

- We can turn this around and write ω_{ij} in terms of Ω. Start with:

$$\Omega_k = -\tfrac{1}{2} e_{ijk} \omega_{ij}$$

- Multiply both sides by e_{mnk}

$$e_{mnk}\Omega_k = -\tfrac{1}{2} e_{mnk} e_{ijk} \omega_{ij}$$

$$e_{mnk}\Omega_k = -\tfrac{1}{2}\left(\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}\right)\omega_{ij}$$

$$e_{mnk}\Omega_k = -\tfrac{1}{2}\left(\delta_{mi}\delta_{nj}\omega_{ij} - \delta_{mj}\delta_{ni}\omega_{ij}\right)$$

$$e_{mnk}\Omega_k = -\tfrac{1}{2}\left(\omega_{mn} - \omega_{nm}\right)$$

$$e_{mnk}\Omega_k = -\tfrac{1}{2}\left(\omega_{mn} + \omega_{mn}\right)$$

$$e_{mnk}\Omega_k = -\omega_{mn}$$

$$\omega_{ij} = -e_{ijk}\Omega_k$$

One Final Manipulation

- Let's go back to the original term in the Taylor Series expansion:

$$u_i^{(rot)} = \omega_{ij} dx_j = -e_{ijk} \Omega_k dx_j = e_{ikj} \Omega_k dx_j$$

$$u^{(rot)} = \Omega \times dx$$

  - This makes it a bit more clear that this is a rotation.

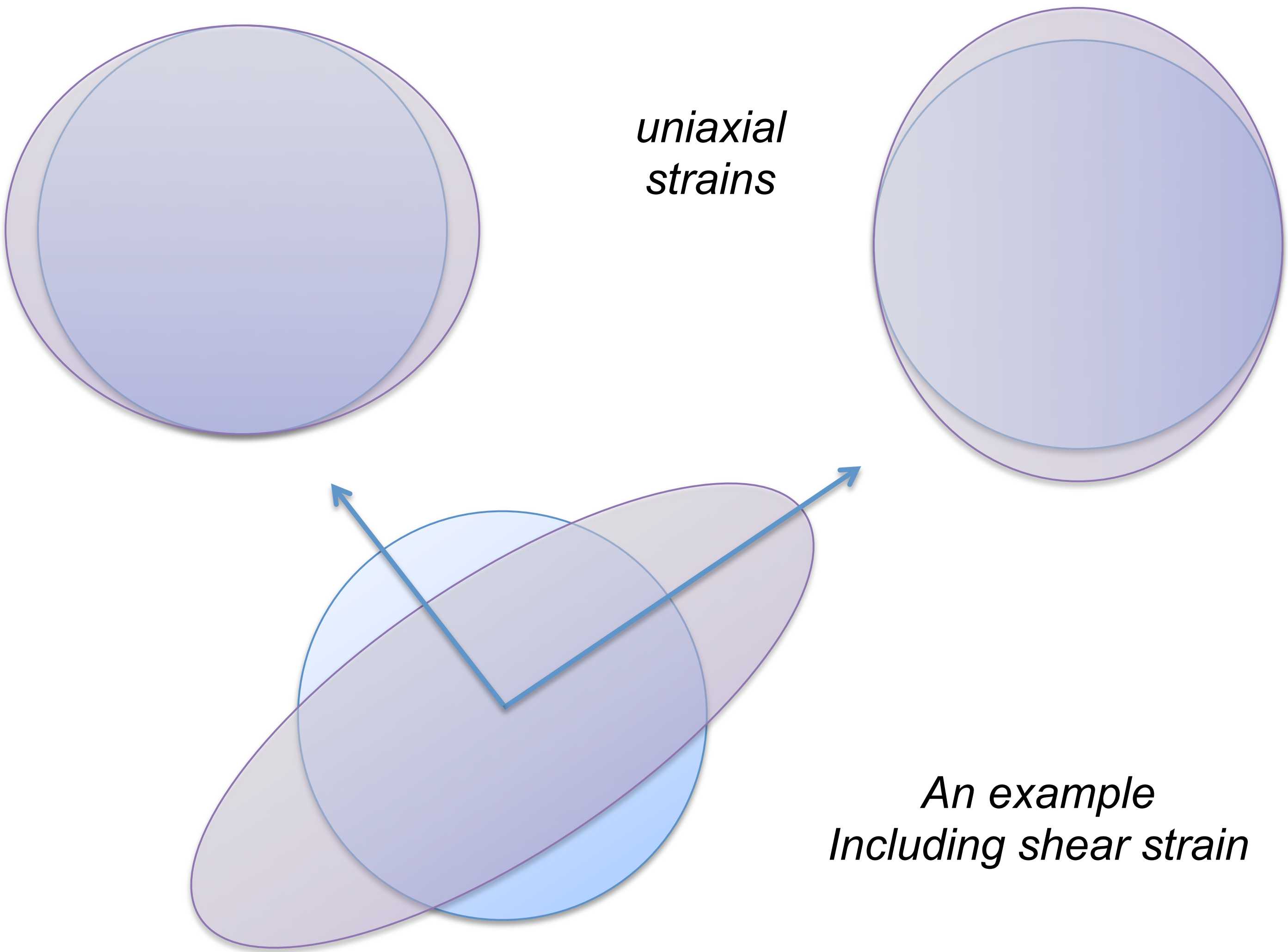
Strain Tensor

Axial Strains

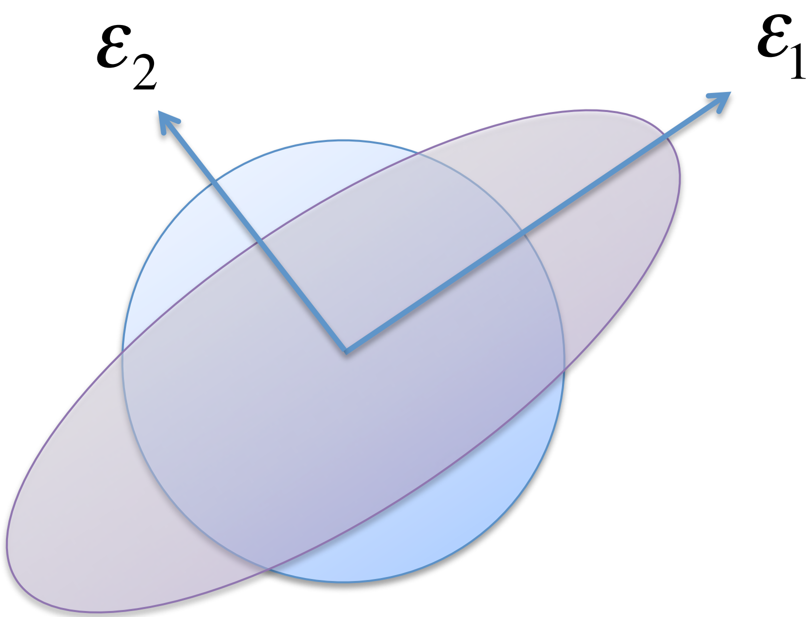
Shear Strains

$$
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$

Examples of Strains



Principal Axes of Strain



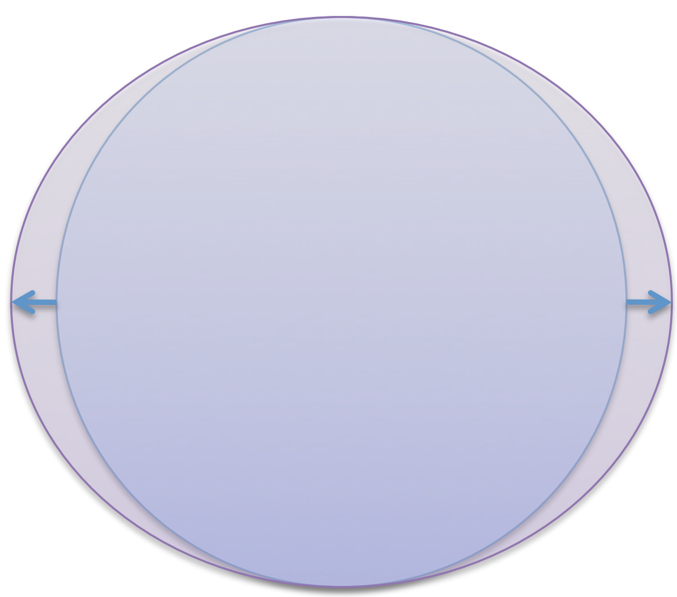
- For any strain, there is a coordinate system where the strains are uniaxial – no shear strain.
- Called the ***principal axes*** of the strain tensor.
  - Strains in those directions are the ***principal strains***.
- Principal axes are the eigenvectors of the strain tensor and principal strains are the eigenvalues.

Strain in a Particular Direction

- Let's look at how we can use the strain tensor to get a line length change in an arbitrary direction. Start with a simple example in 2D:

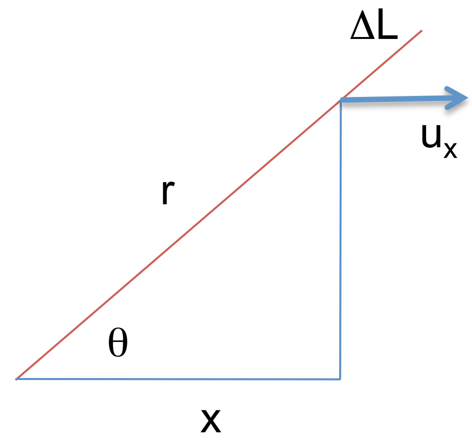
$$\begin{bmatrix} \varepsilon_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \varepsilon_{11} > 0$$

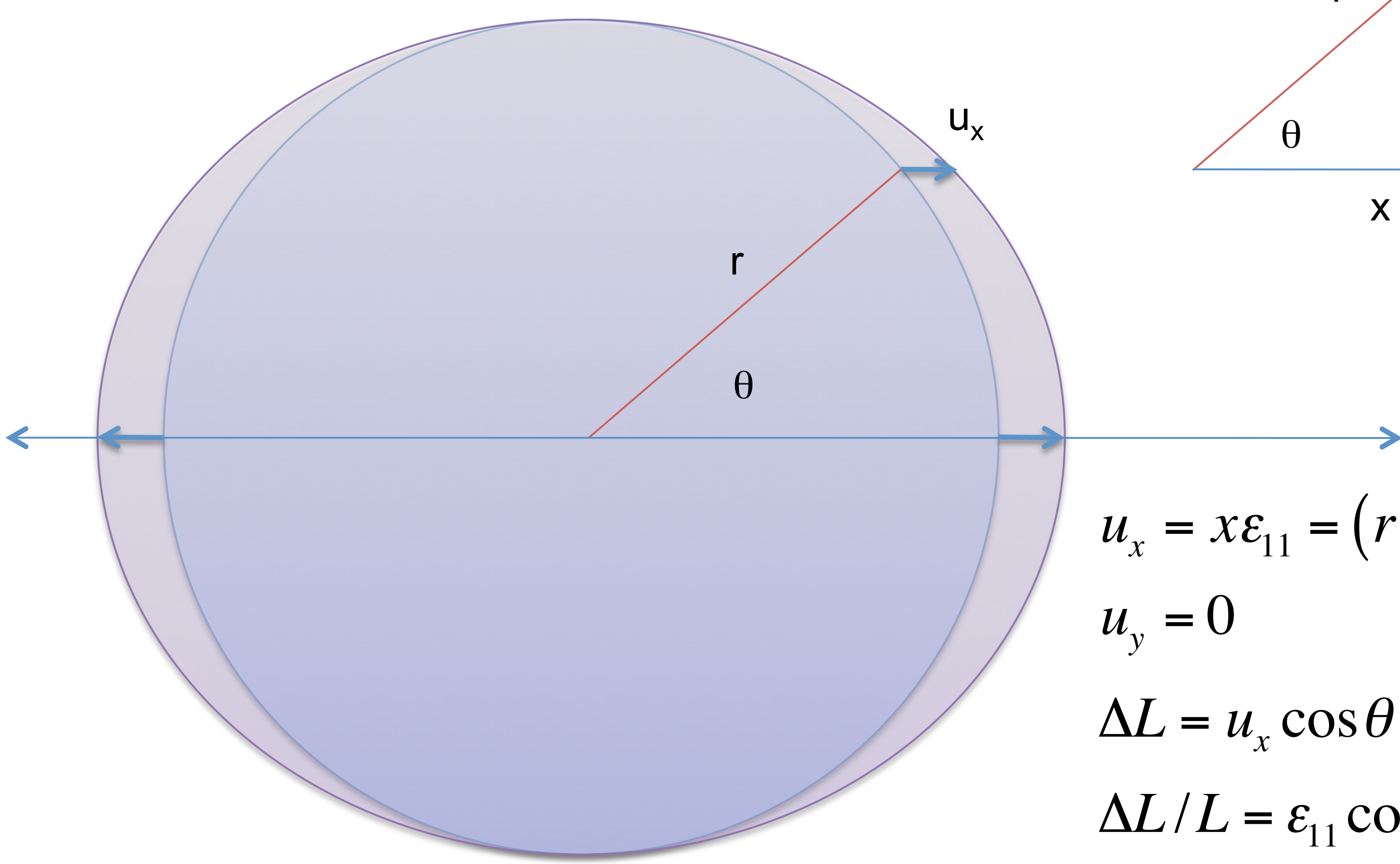
- If we start with a circle, it will be deformed into an ellipse.



Strain in a Particular Direction

$$\begin{bmatrix} \varepsilon_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \varepsilon_{11} > 0$$





$$u_x = x\varepsilon_{11} = (r \cdot \cos\theta)\varepsilon_{11}$$

$$u_y = 0$$

$$\Delta L = u_x \cos\theta = r\varepsilon_{11}\cos^2\theta$$

$$\Delta L / L = \varepsilon_{11}\cos^2\theta$$

Strain in a Particular Direction

- For a strain only in the x direction,

$$\begin{bmatrix} \varepsilon_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \varepsilon_{11} > 0 \quad \Delta L / L = \varepsilon_{11} \cos^2 \theta$$

- If we do the same for a strain in only in the y direction, we get something similar,

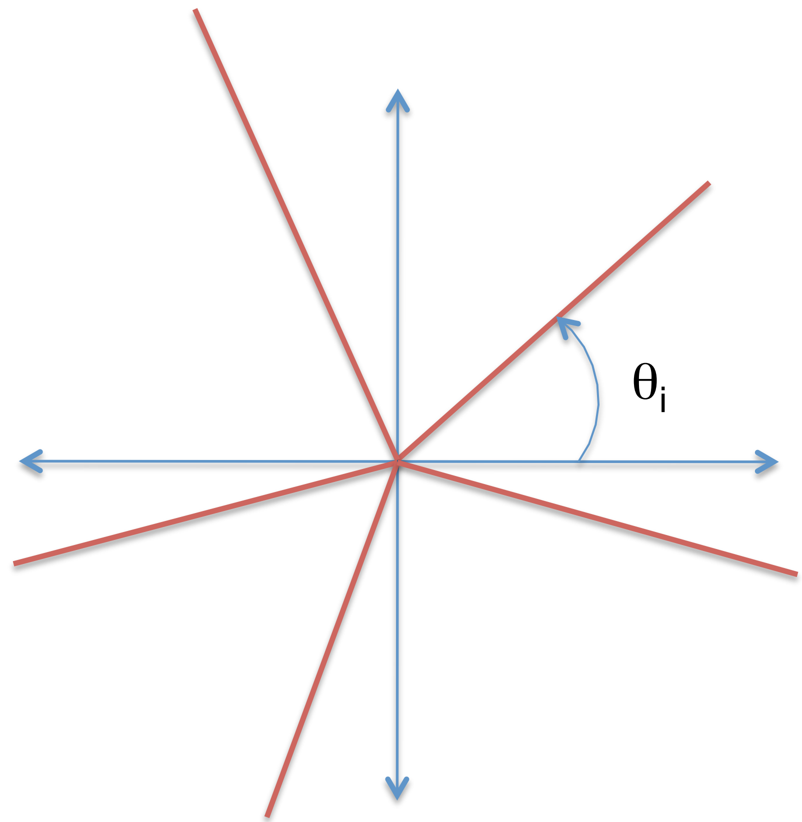
$$\begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_{22} \end{bmatrix} \quad \varepsilon_{22} > 0 \quad \Delta L / L = \varepsilon_{22} \sin^2 \theta$$

- In general, we get an equation for an ellipse:

$$e_{rr} = \varepsilon_{11} \cos^2 \theta + \varepsilon_{22} \sin^2 \theta + \varepsilon_{12} \sin 2\theta$$

Suppose we have multiple lines

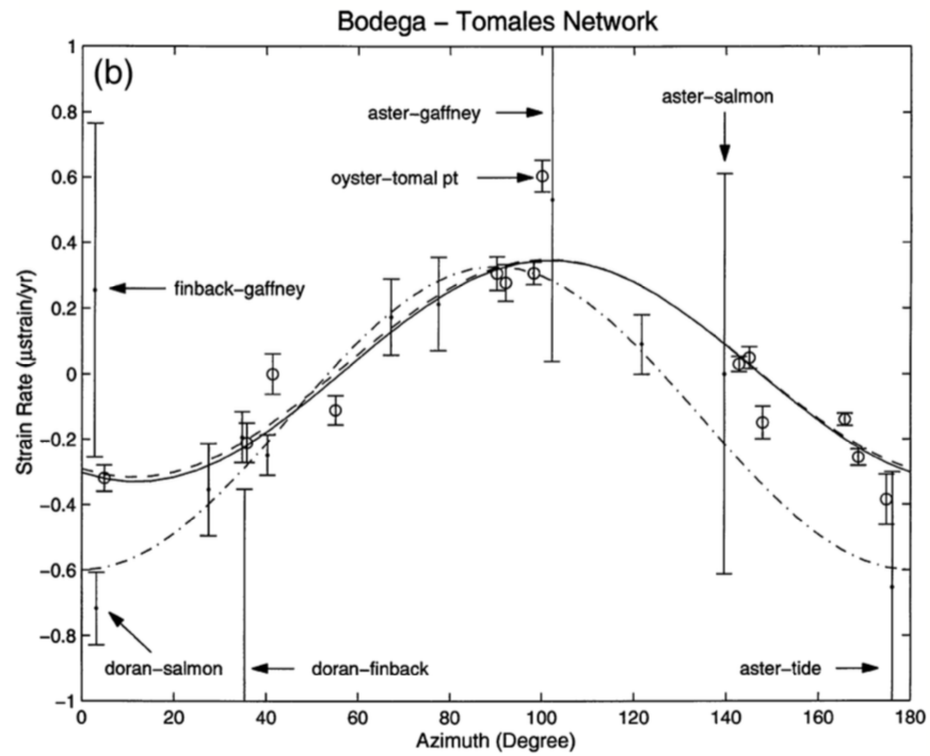
- If we measure line length changes for several lines of different orientations, θ_i
- If we have lines of three or more different orientations, we can estimate the three horizontal strains.



$$\left(e_{rr}\right)_i = \varepsilon_{11}\cos^2\theta_i + \varepsilon_{22}\sin^2\theta_i + \varepsilon_{12}\sin 2\theta_i$$

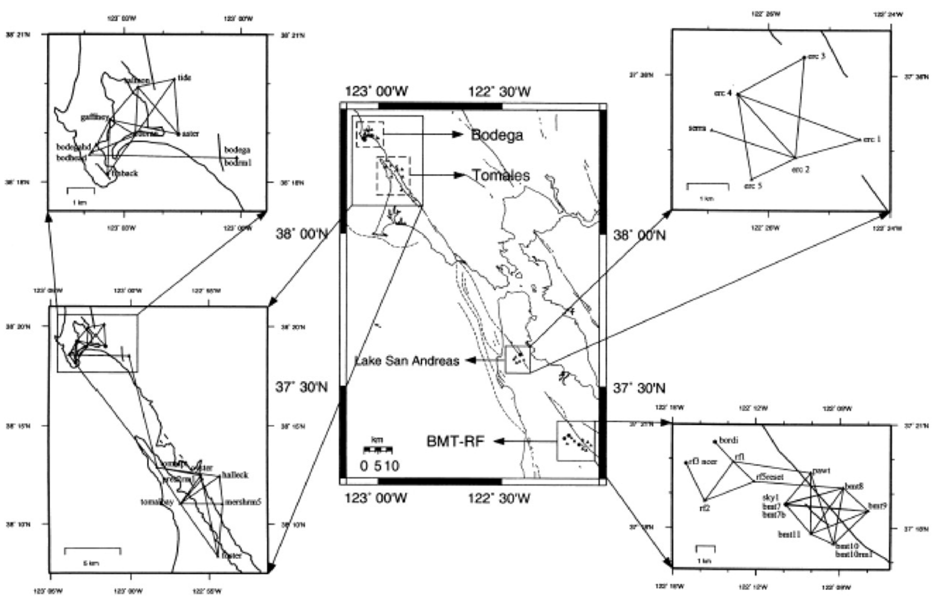
Suppose we have multiple lines

- This is the basis for estimating strain from EDM networks
- The lines don't have to have the same origin. I just drew it that way for convenience.
- The more lines of different orientations, the better the estimate becomes



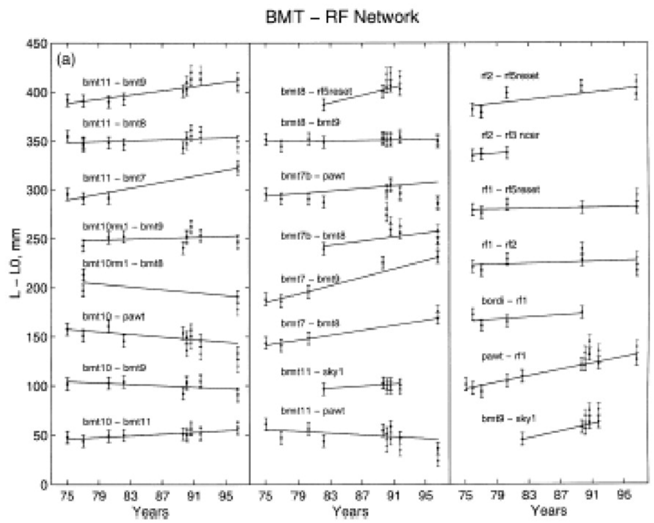
$$\left(e_{rr}\right)_i = \varepsilon_{11}\cos^2\theta_i + \varepsilon_{22}\sin^2\theta_i + \varepsilon_{12}\sin 2\theta_i$$

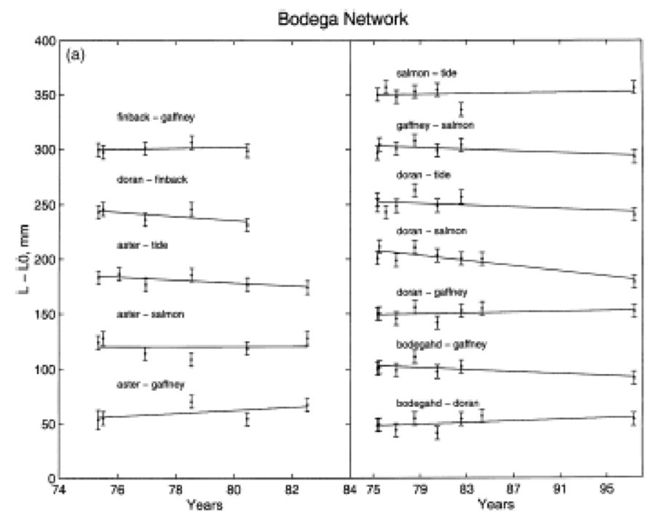
Chen and Freymueller (2002)

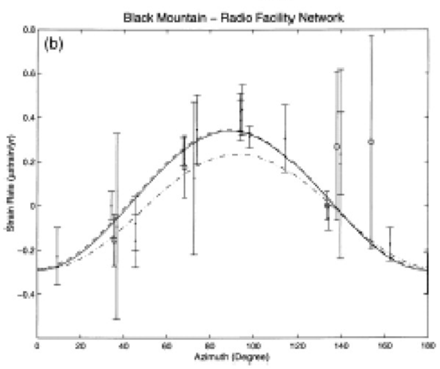


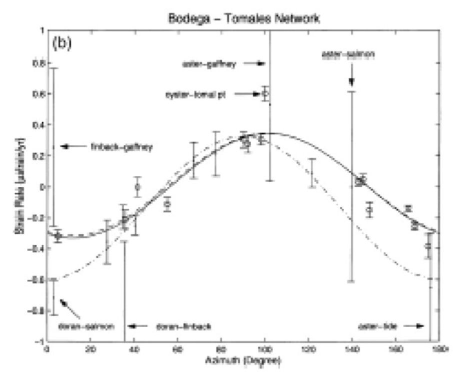
- Evaluated strain rates for four different near-fault EDM networks along the San Andreas fault
- All sites were so close to the fault we assume uniform strain within the network

Estimated Strain Rate Tensors







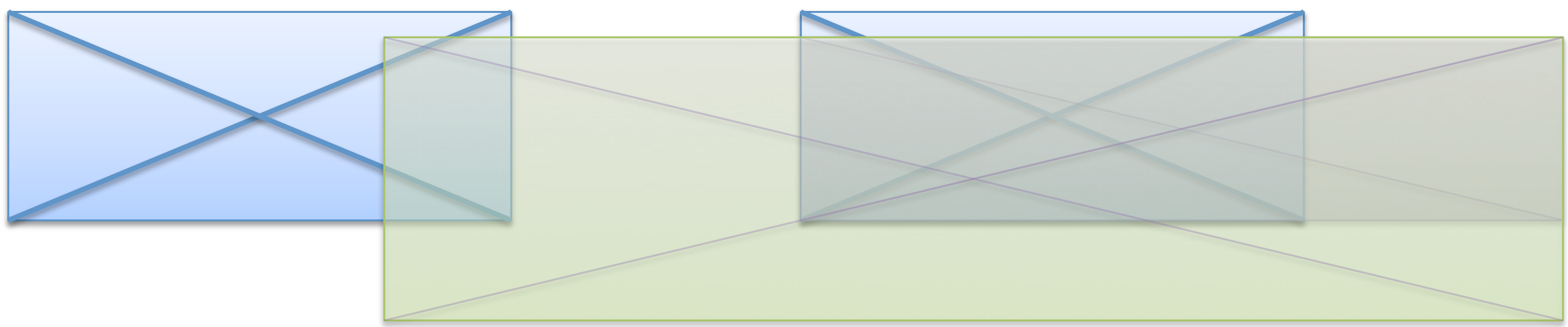


Estimating Strain from Line Lengths

- Obviously, what we looked at is a 2D problem. We assumed the state of ***plane strain***, in which all deformation is horizontal.
- Line lengths or line length changes are invariant under rotations, so we can learn nothing about the rotation tensor from them
  - But we also can estimate strain without worrying about rotation
- We have to assume uniform strain, which might be the wrong assumption.
- An alternative might be to assume simple shear (no axial strains), such as for glacial flow or some strike-slip fault motion problems.

Strain and Angle Changes

- If we strain an object, lines within the object will, in general, rotate:



- But a uniform strain (uniform scaling up or down in size) does not rotate any lines. So that means we can't estimate all three components of the strain tensor from angle changes.

Estimating Strain from Angle Changes

- We can't estimate the dilatation (Δ). Rewrite our three horizontal strains this way:

$$\Delta = \varepsilon_{11} + \varepsilon_{22}$$ *Dilatation*

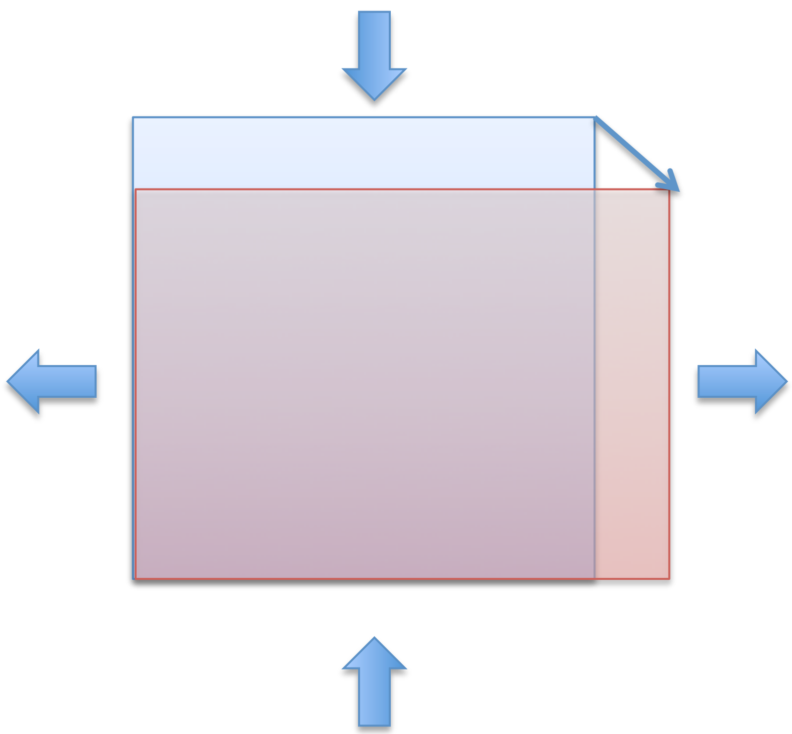
$$\gamma_1 = \varepsilon_{11} - \varepsilon_{22}$$ *Pure shear with N-S contraction, E-W extension*

$$\gamma_2 = 2\varepsilon_{12}$$ *Pure shear with NW-SE contraction, NE-SW extension (or simple shear with displacement along x-axis)*

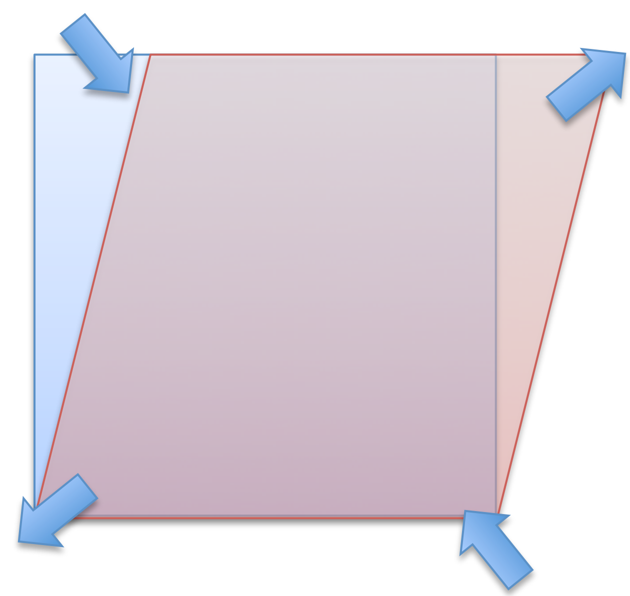
- The gammas are called the ***engineering shear strains*** (the epsilons are the tensor strains). The factor of 2 is from historical practice.
  - Older papers often use the gammas. Newer work usually uses tensor strains. Be careful of the factor of 2!

Illustrations of Engineering Shears

$\gamma_1 = \varepsilon_{11} - \varepsilon_{22} > 0$



$\gamma_2 = 2\varepsilon_{12} > 0$



What Can Be Determined From the Data?

- There is a general lesson here:
  - You might want to know all components of the strain and rotation tensors.
  - When your data only determine some parts of your parameters, it makes sense to re-parameterize the problem so that you directly estimate what your data can determine
    - The alternative is to put possibly arbitrary constraints on your parameters
  - The discussion of what you can learn about a plate rotation from a single site was another example.
  - What we are doing is breaking up our parameters into two sets (determined and impossible to determine) that are (ideally) orthogonal in some way.

More examples of this

- If you have GPS displacement/velocity data, you can determine all components of the displacement gradient tensor (or strain + rotation)
- If you have line length changes, you can determine all components of strain, but nothing about rotation
  - So don't try to estimate the displacement gradient tensor; estimate the strain tensor instead
- If you have angle changes, you can't determine rotation or dilatation
  - So don't try to estimate the strain tensor; estimate the gammas instead.

Axis of Maximum Shear

- One other thing you can get out of the gammas is the axis of maximum shear

$$\tan 2\Theta_s = \gamma_1 / \gamma_2$$

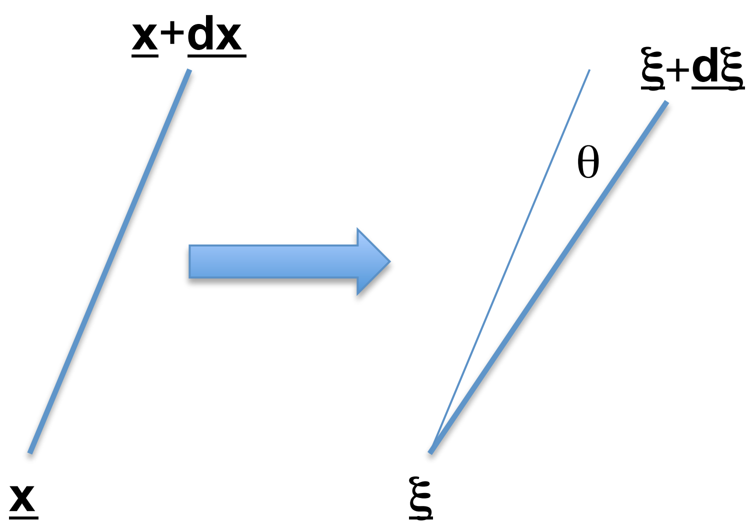
$$\Theta_s = \tfrac{1}{2}\tan^{-1}(\gamma_1/\gamma_2)$$

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$$

Magnitude of maximum shear

  - If $\gamma_1 = 0$, the maximum shear is along the x-axis.
  - If $\gamma_2 = 0$, the maximum shear is 45° from the x-axis.

(Quantitative) Rotation of a Line Segment



Define the rotation vector Θ

$$\Theta = \frac{dx \times d\xi}{|dx| \cdot |d\xi|}$$

$$|\Theta| = \sin\theta \approx \theta$$

- Define the rotation vector of the line by the equation at left. In index notation it is:

$$\Theta_i = \frac{e_{ijk} dx_j d\xi_k}{dx_n dx_n}$$

- Because $\xi_i = x_i + u_i$,

$$d\xi_k = \frac{\partial \xi_k}{\partial x_m} dx_m = \left(\delta_{km} + \frac{\partial u_k}{\partial x_m} \right) dx_m$$

- Note: curly d and straight d are not the same (partial vs. total derivatives).
- Also the magnitudes of d**x** and d**ξ** are equal for a pure rotation.

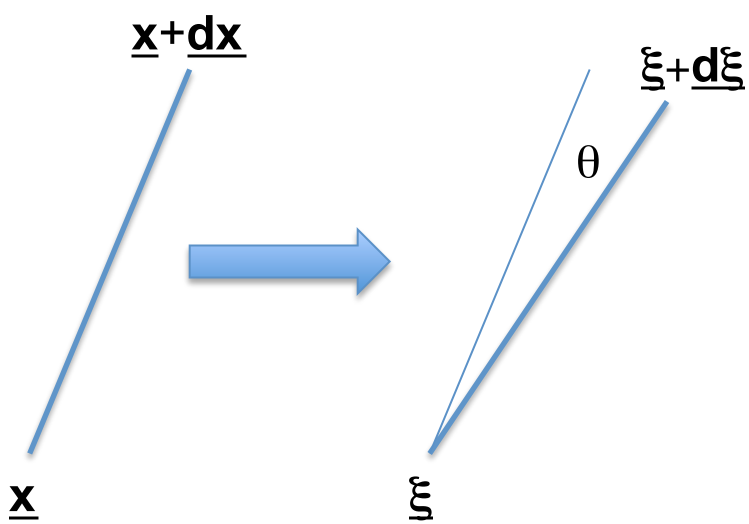
Total vs. Partial Derivative

- There is a subtlety to partial derivatives that you might or might not know.
- Assume a function of space and time, f(x,y,t), where the variables x and y also depend on time.
  - Example: A function that depends on the angle of the sun, which depends on your position (as a function of time if you are moving), and with time directly
  - The function may have a direct dependence on time, and an indirect dependence that comes from x(t) and y(t).
- The ***total derivative*** is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \sum_{j=1}^{k} \frac{\mathrm{d}y_j}{\mathrm{d}x}\frac{\partial}{\partial y_j},$$

(Quantitative) Rotation of a Line Segment



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- Note: curly d and straight d are not the same (partial vs. total derivatives).
- Also the magnitudes of d**x** and dξ are equal for a pure rotation.

Rotation of Line Segment

- Substituting this in gives:

$$\Theta_i = \frac{1}{dx_n dx_n}\left[e_{ijk} dx_j \left(\delta_{km} + \frac{\partial u_k}{\partial x_m}\right)\right] dx_m$$

$$\Theta_i = \frac{1}{dx_n dx_n}\left[e_{ijk} dx_j \left(\delta_{km} dx_m + \frac{\partial u_k}{\partial x_m} dx_m\right)\right]$$

$$\Theta_i = \frac{1}{dx_n dx_n}\left[e_{ijk} dx_j dx_k + e_{ijk} dx_j \frac{\partial u_k}{\partial x_m} dx_m\right]$$

This term is just **x** x **x** = 0

This term is the velocity gradient tensor, or strain + rotation

$$\Theta_i = \frac{dx_j dx_m}{dx_n dx_n}\left[0 + e_{ijk}\left(\varepsilon_{km} + \omega_{km}\right)\right]$$

These are just unit vectors

$$\Theta_i = e_{ijk}\left(\varepsilon_{km} + \omega_{km}\right) d\hat{x}_j d\hat{x}_m$$

What Does This Mean?

- Here's the equation:

$$\Theta_i = e_{ijk}\left(\varepsilon_{km} + \omega_{km}\right)d\hat{x}_j d\hat{x}_m$$

- This equation tells us that a line segment can rotate because of either a strain or a rotation.
  - The rotation part is obvious
  - The strain part we saw earlier (graphically). Depending on the values of the strain components, the rotation might be zero or non-zero.
  - Also, the two terms might cancel out!