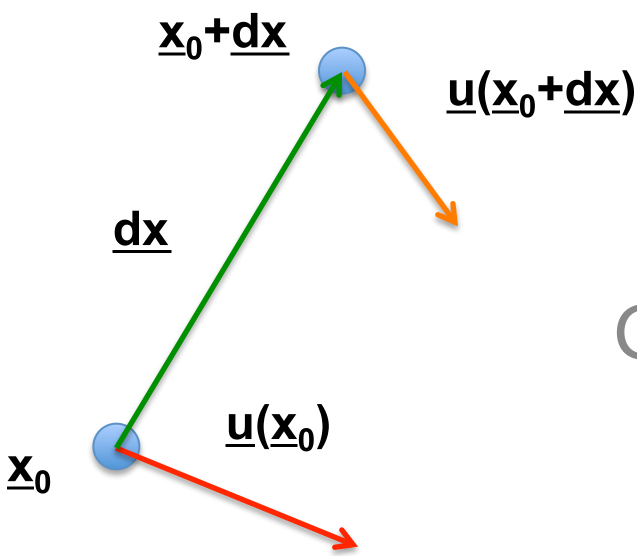
Lecture 12: Coordinates, Transformations, vectors, displacements, strains

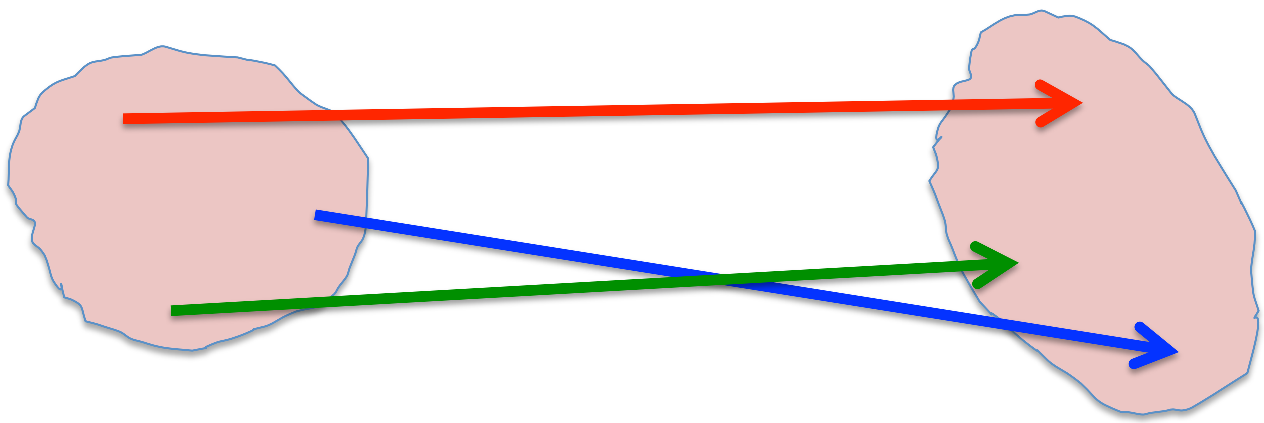


GEOS 655 Tectonic Geodesy
Jeff Freymueller

Outline

- Coordinates and Transformations
  - The mathematical basis for the problem
  - In particular, how to do rotations
- Motions in general
  - displacement = rigid body motion + deformation
  - displacement = translation + rotation + deformation
- Deformation = strain

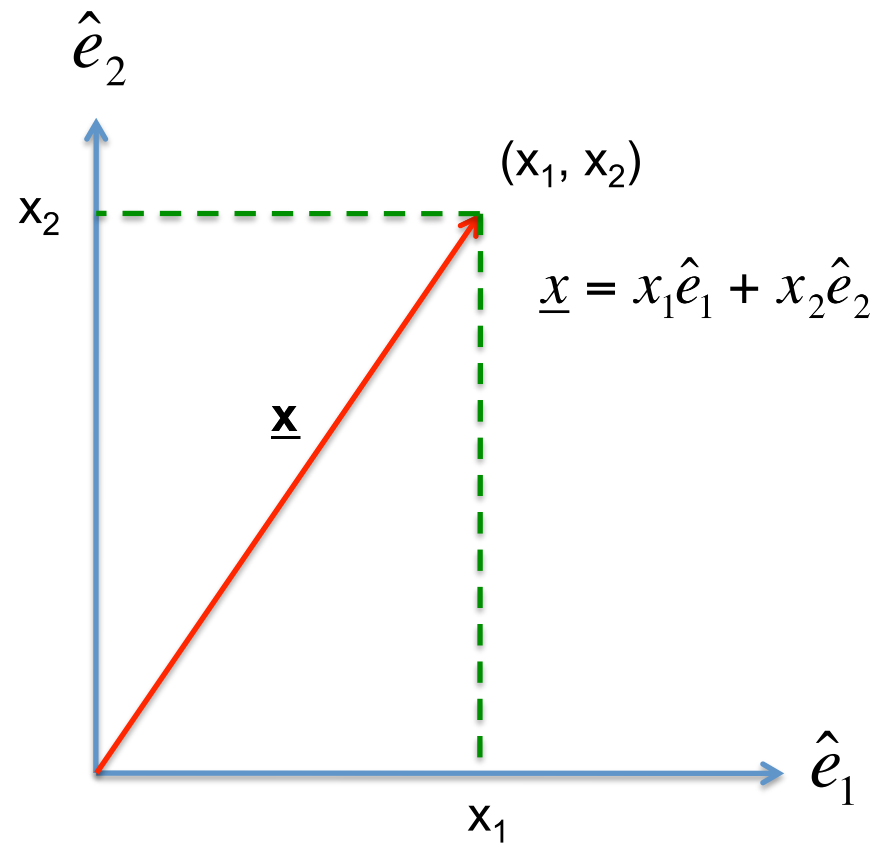
Motion in General



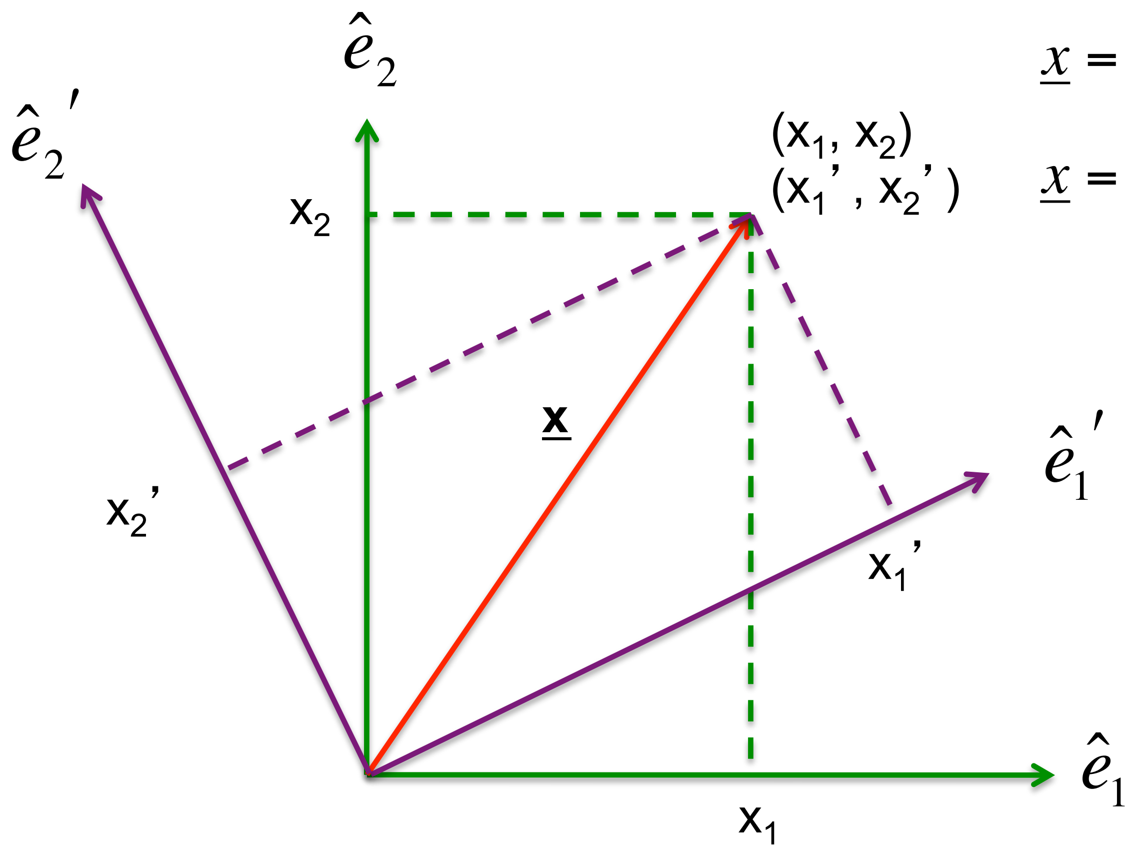
– displacement = rigid body motion + deformation
– displacement = translation + rotation + deformation

Coordinates and Vectors (in 2D)

- Given a point and coordinate axes, we can define a pair of ***coordinates*** that locate the point.
  - Vector may be a physical thing
  - Coordinates represent the vector
  - Imagine a vector that points from the center of the earth to a point on the surface. If we change coordinate systems, the vector still points to the same place, but the coordinates or vector components will be different.



Same Vector, Different Coordinates



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

$$\underline{x} = x_1{}'\hat{e}_1{}' + x_2{}'\hat{e}_2{}'$$

Mathematically, we often blur the distinction between the vector and its representation.

Many Different Vector Notations

$$\underline{x} = (x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

row vector

column vector

- This leads to a variety of notations for vector operations, for example the dot product:

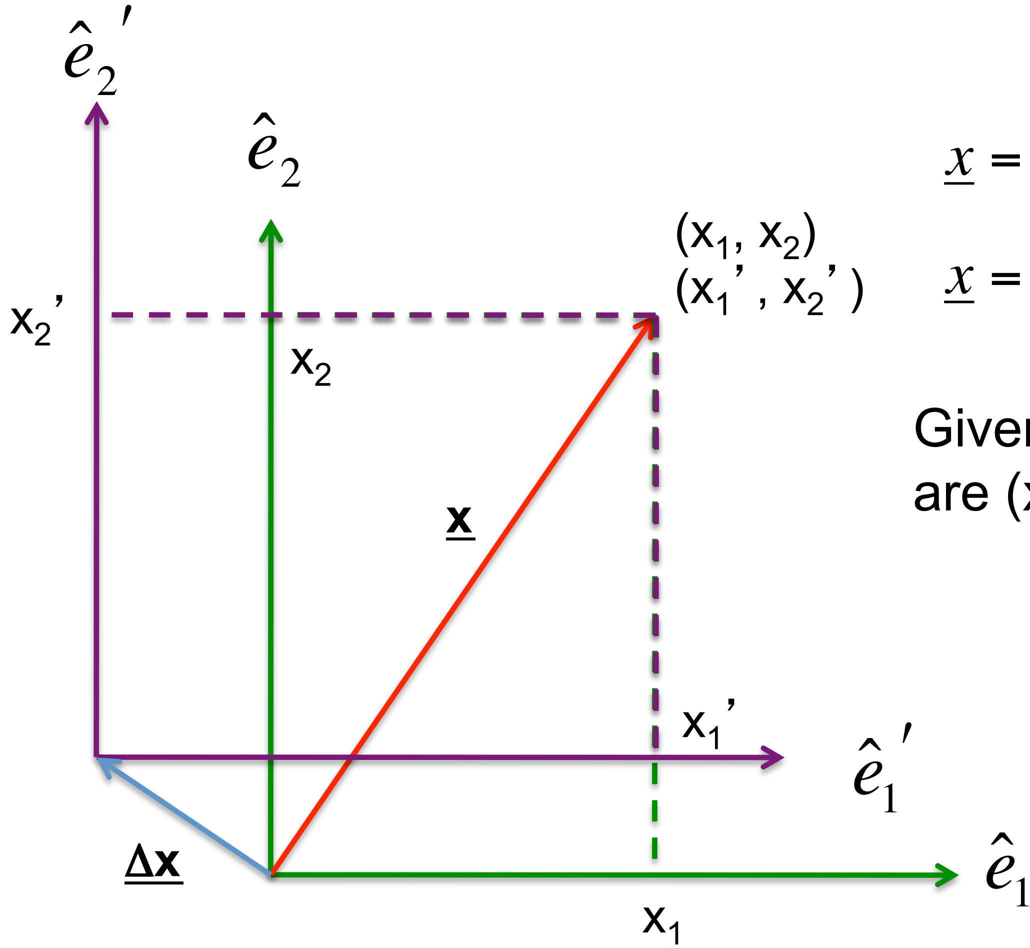
$$\begin{aligned} \underline{x} \cdot \underline{y} &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= \sum_{i=1}^{3} x_i y_i \\ &= x_i y_i \\ &= \underline{x}^T \underline{y} \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

"Einstein convention": summation over repeated indices

Transformations

- Mathematical ***transformations*** map something from one system (or set) to another
  - For example, transform the coordinates of a vector from one coordinate system to another
  - A general form: $\underline{\mathbf{y}} = T\underline{x}$
- Linear transformations are part of the basis of ***linear algebra***.
  - A mathematician would say that a transformation is a mapping from one "space" to another.
- We can represent linear transformations in a variety of ways, but the two most common are a matrix representation or using index notation

Translational Transformations



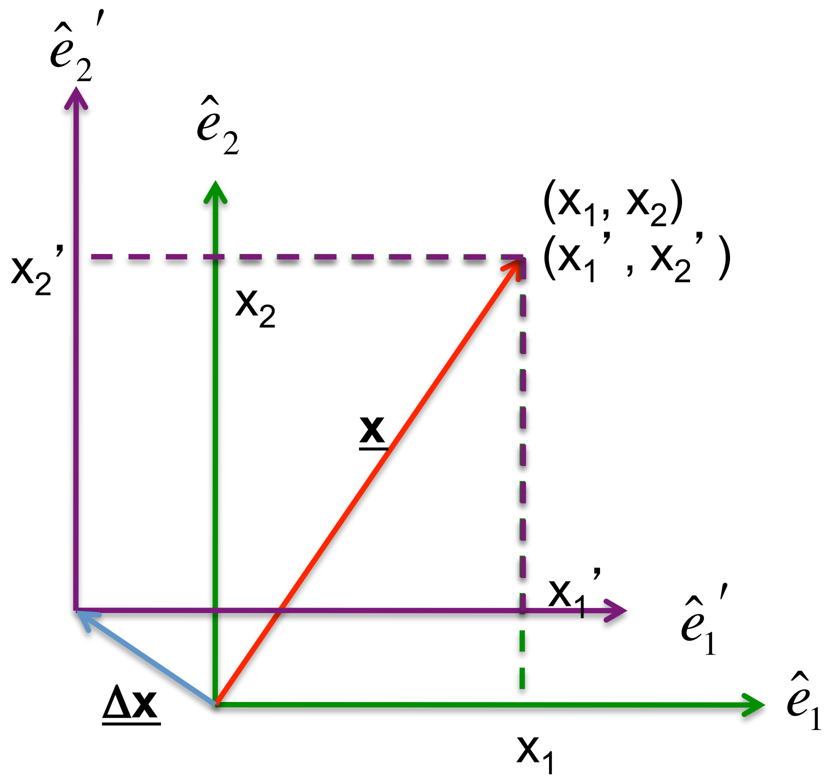
$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

$$\underline{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

Given (x_1, x_2), what are (x_1', x_2')?

In this case the unit vectors describing the coordinate system are the same, only the origin of the system is different.

Translational Transformations



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

$$\underline{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

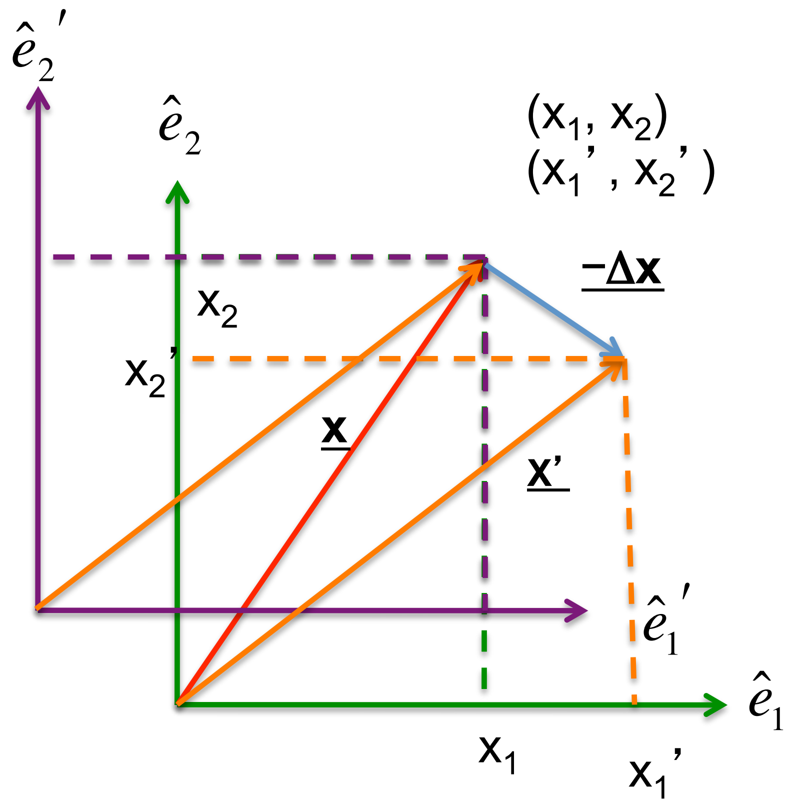
Given (x_1, x_2), what are (x_1', x_2')?

This is pretty easy:

$$x_1' = x_1 - \Delta x_1$$

$$x_2' = x_2 - \Delta x_2$$

Translational Transformations



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

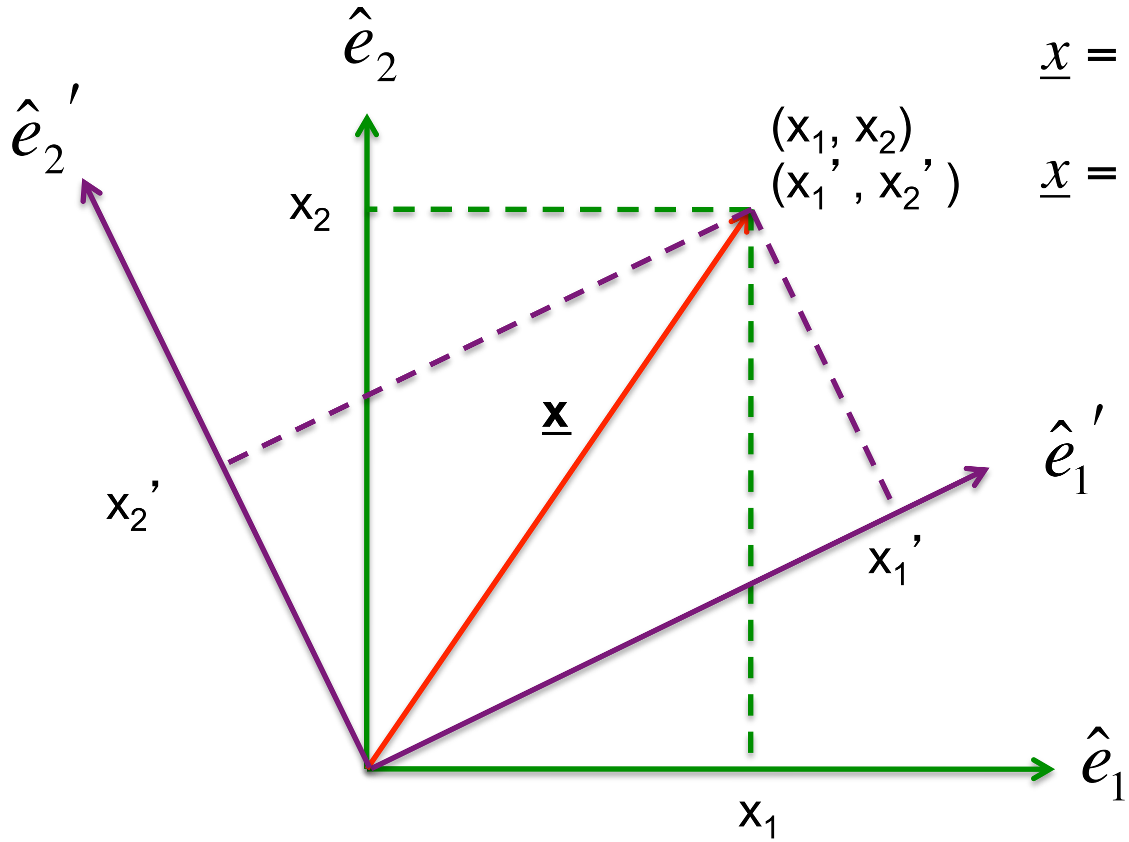
$$\underline{x}' = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

$$x_1' = x_1 - \Delta x_1$$

$$x_2' = x_2 - \Delta x_2$$

This vector $\underline{\mathbf{x}}'$ is also given by $\underline{\mathbf{x}}' = \underline{\mathbf{x}} - \underline{\Delta\mathbf{x}}$
Translating the coordinate system by $\underline{\Delta\mathbf{x}}$ is like subtracting $\underline{\Delta\mathbf{x}}$ from the vector.

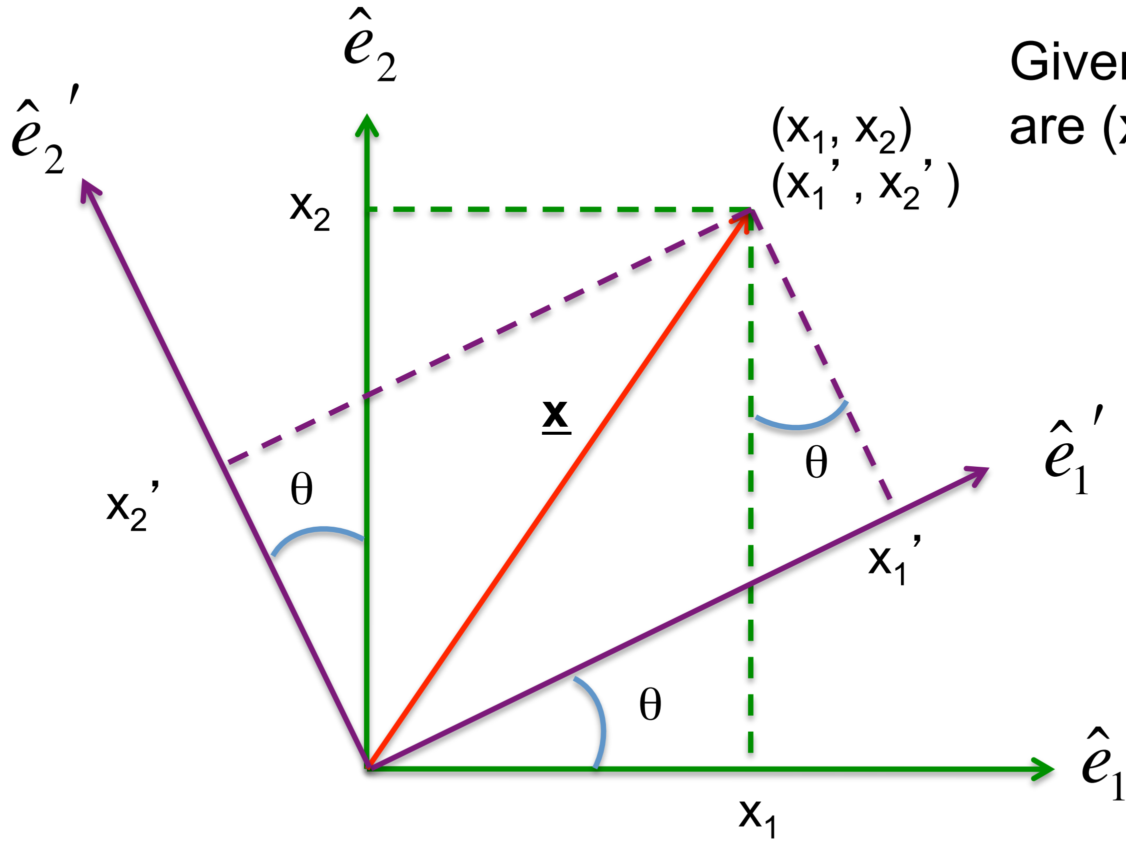
Rotational Transformations



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

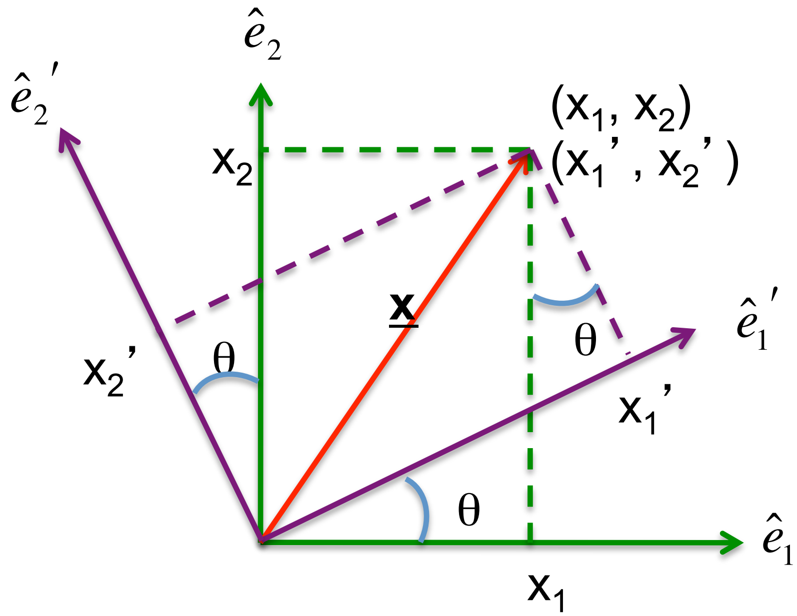
$$\underline{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

Given two systems, how are the components of a vector related?



Given (x_1, x_2), what are (x_1', x_2')?

Given two systems, how are the components of a vector related?

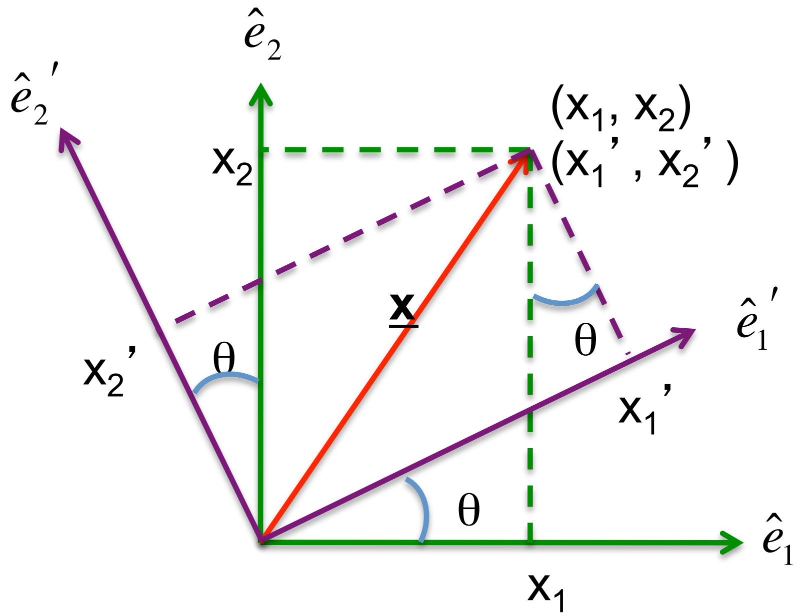


Given (x_1, x_2), what are (x_1', x_2')?

$$x_1 = x_1' \cos\theta - x_2' \sin\theta \qquad x_1' = x_1 \cos\theta + x_2 \sin\theta$$

$$x_2 = x_1' \sin\theta + x_2' \cos\theta \qquad x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

Given two systems, how are the components of a vector related?



Given (x_1, x_2), what are (x_1', x_2')?

$$x_i = \beta_{ij} x_j'$$ *implied summation*

$$\left(\beta_{ij}\right) = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Transformation and Inverse

- The matrix of the betas is the ***rotation matrix***, which maps coordinates in the primed system into coordinates in the unprimed system:

$$x_i = \beta_{ij} x_j'$$

- The ***inverse transformation*** (opposite rotation) goes the other way, from the unprimed to the primed system

$$x_i' = \beta_{ji} x_j = \beta_{ki} x_k$$

$$\left(\beta_{ji}\right) = \left(\beta_{ij}\right)^T = \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Transformation and Inverse

- Combining the two equations:

$$x_i = \beta_{ji} x_j{}' = \beta_{ji}\left(\beta_{jk} x_k\right) = \left(\beta_{ji} \cdot \beta_{jk}\right) x_k$$

$$\left(\beta_{ji} \cdot \beta_{jk}\right) = \delta_{ik}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix}$$

matrix form →

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} \cdot \left(\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$\begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = I$$

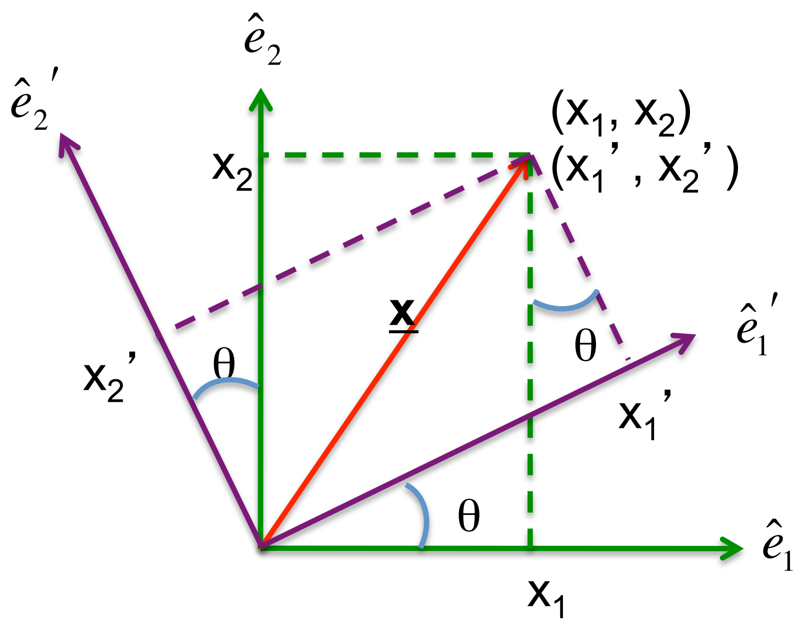
Transformation and Inverse

- The transpose of the rotation matrix is its inverse.

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}^T = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}^{-1}$$

- This kind of matrix is called an ***orthogonal matrix***.
- Rotational transformations preserve length
  - The length of the vector is the same in both coordinate systems.
  - Other kinds of linear transformations can be done that do not preserve length.

Geometric Relationships



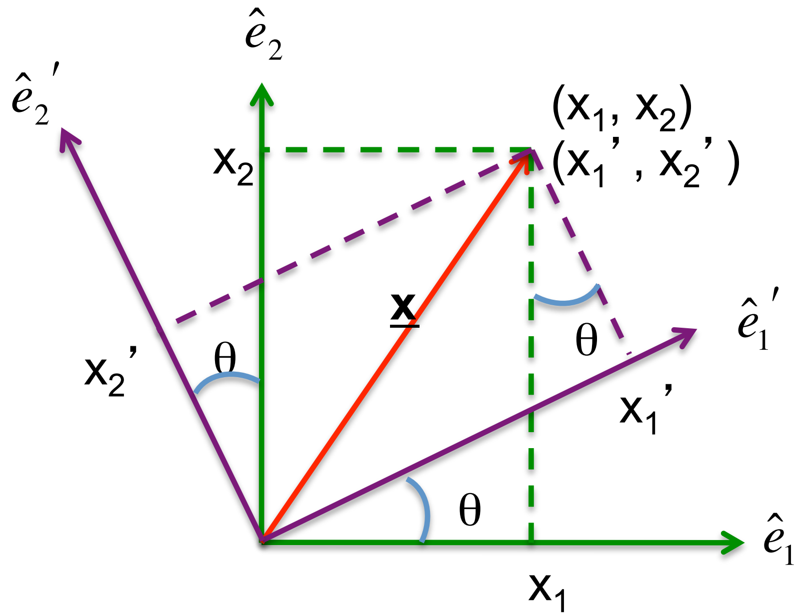
$$\beta_{11} = \cos\theta = \hat{e}_1 \cdot \hat{e}_1'$$

$$\beta_{ij} = \hat{e}_i \cdot \hat{e}_j'$$

- The betas have an easy geometric interpretation
  - Sometimes called ***direction cosines***

$$\left(\beta_{ij}\right) = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Geometric Relationships



$$\underline{x} = x_j \hat{e}_j = x_j' \hat{e}_j'$$

- Take the dot product of both sides with e_i

$$\left(x_j \hat{e}_j\right) \cdot \hat{e}_i = \left(x_j' \hat{e}_j'\right) \cdot \hat{e}_i$$

$$x_j \left(\hat{e}_j \cdot \hat{e}_i\right) = x_j' \left(\hat{e}_j' \cdot \hat{e}_i\right)$$

$$\delta_{ij} x_j = \left(\hat{e}_i \cdot \hat{e}_j'\right) x_j'$$

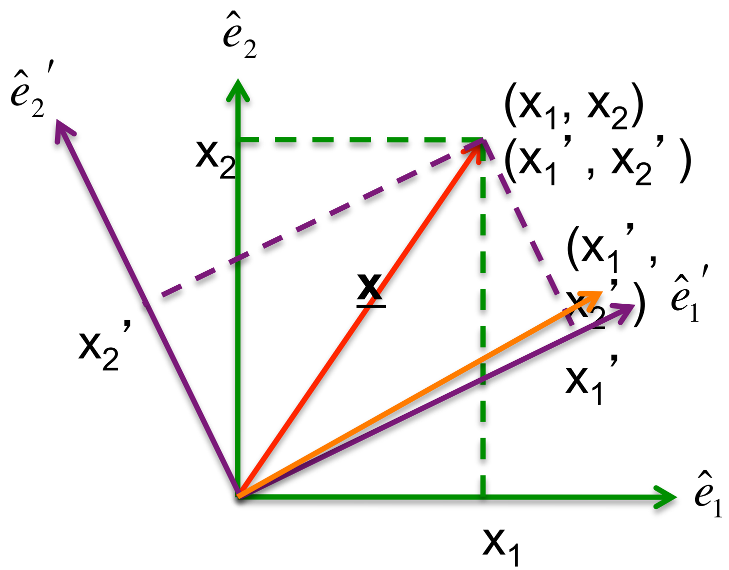
$$x_i = \left(\hat{e}_i \cdot \hat{e}_j'\right) x_j' = \beta_{ij} x_j'$$

Rotating Cooordinate System vs. Rotating the Vector

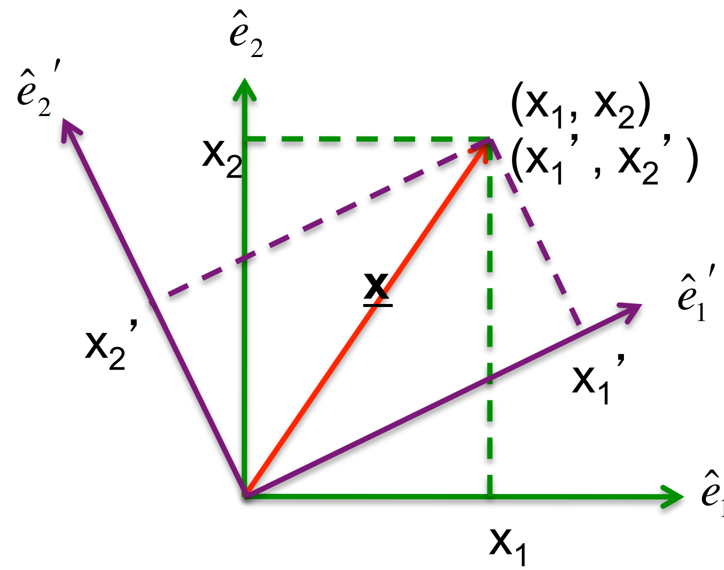
Rotating a vector is the same as rotating the coordinate system in the opposite direction

$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

$$\underline{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

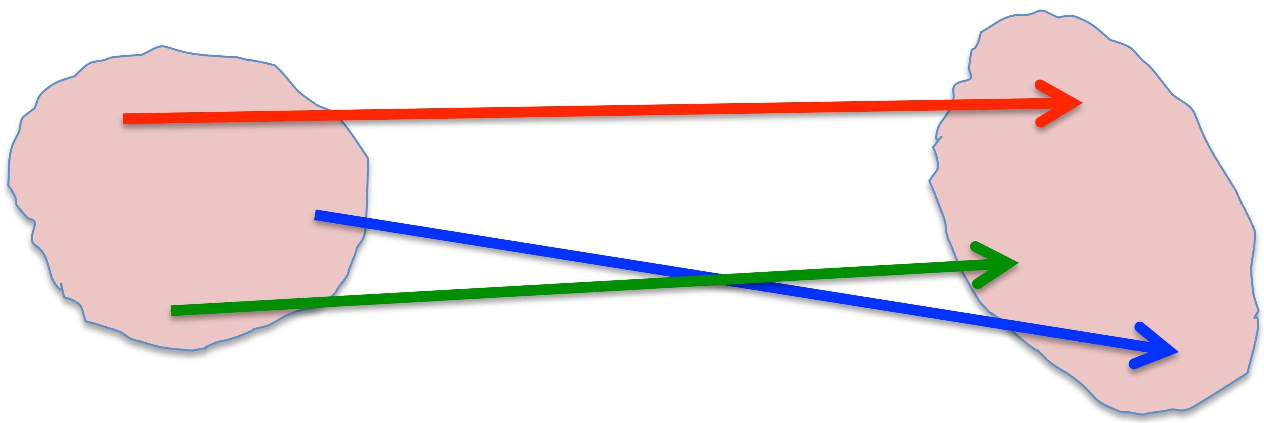


Clockwise rotation of vector



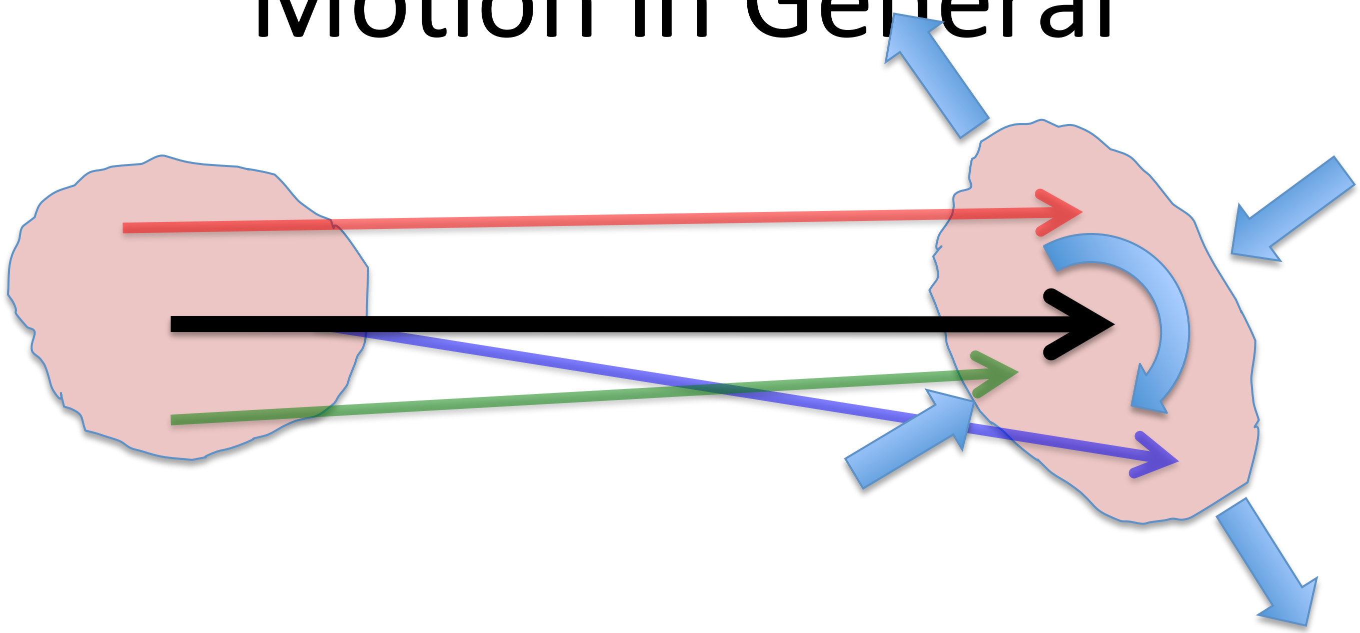
Counter-clockwise rotation of coordinate system

Motion in General



- displacement = rigid body motion + deformation
- displacement = translation + rotation + deformation

Motion in General



- displacement = rigid body motion + deformation
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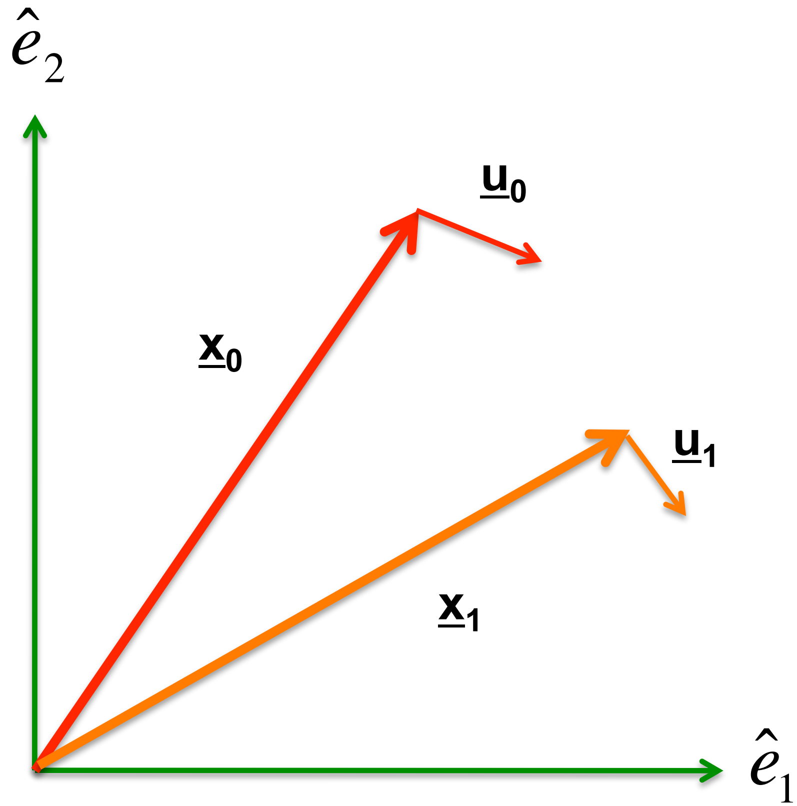
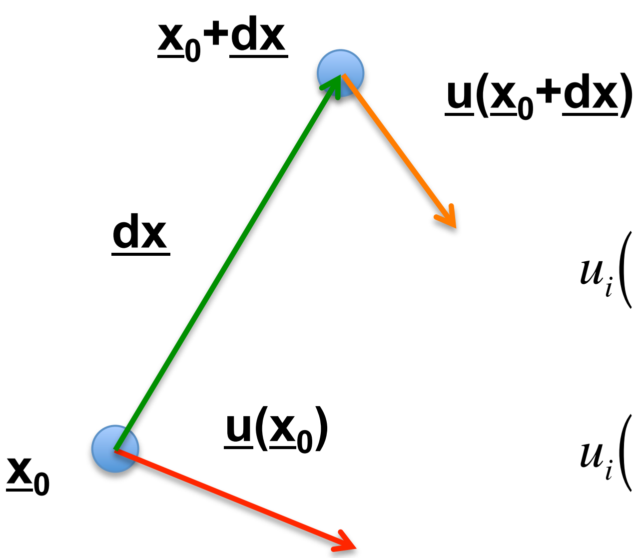
Displacement



- Every point has some displacement
- $\underline{\mathbf{u}}(x_1,x_2,x_3) = \underline{\mathbf{u}}(\underline{\mathbf{x}})$
- How do we differentiate between rigid motion and deformation?
  - Consider the motion of neighboring points

Displacement



- Use a Taylor Series expansion to relate the two displacements:

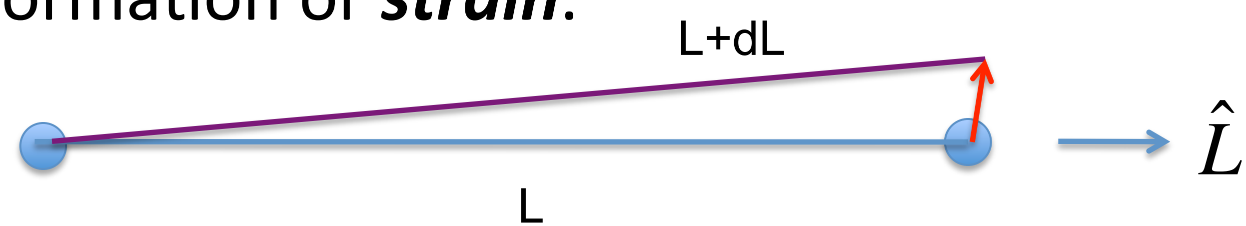
$$u_i(\underline{x}_0 + \underline{dx}) = u_i(\underline{x}_0) + \left(\frac{\partial u_i}{\partial x_j}\right)_{x=x_0} dx_j + \cdots$$

$$u_i(\underline{x}_0 + \underline{dx}) = u_i(\underline{x}_0) + \left(\frac{\partial u_i}{\partial x_1}\right)dx_1 + \left(\frac{\partial u_i}{\partial x_2}\right)dx_2 + \left(\frac{\partial u_i}{\partial x_3}\right)dx_3$$

This is a set of 3 equations, for i = 1, 2, 3
First term: translation
Remaining terms: rotation + strain
There are 9 values (du_i/dx_j) (i=1,3; j=1,3)

First a More Physical Point of View

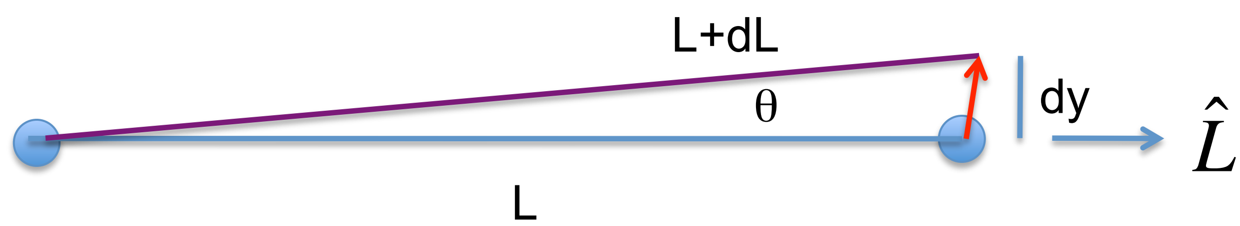
- Let's first look at strain from the point of view of line length. We already know that rotations don't change line lengths, so any line length change comes from deformation or ***strain***.



- If one point is displaced relative to another, the line length between them changes from L to L+dL
  - Assume dL << L (for tectonics: L~km, dL~cm)
- Define the strain in the direction of the line to be the fractional change in length, or strain $\varepsilon_{LL} = (dL/L)$

Shear Strains

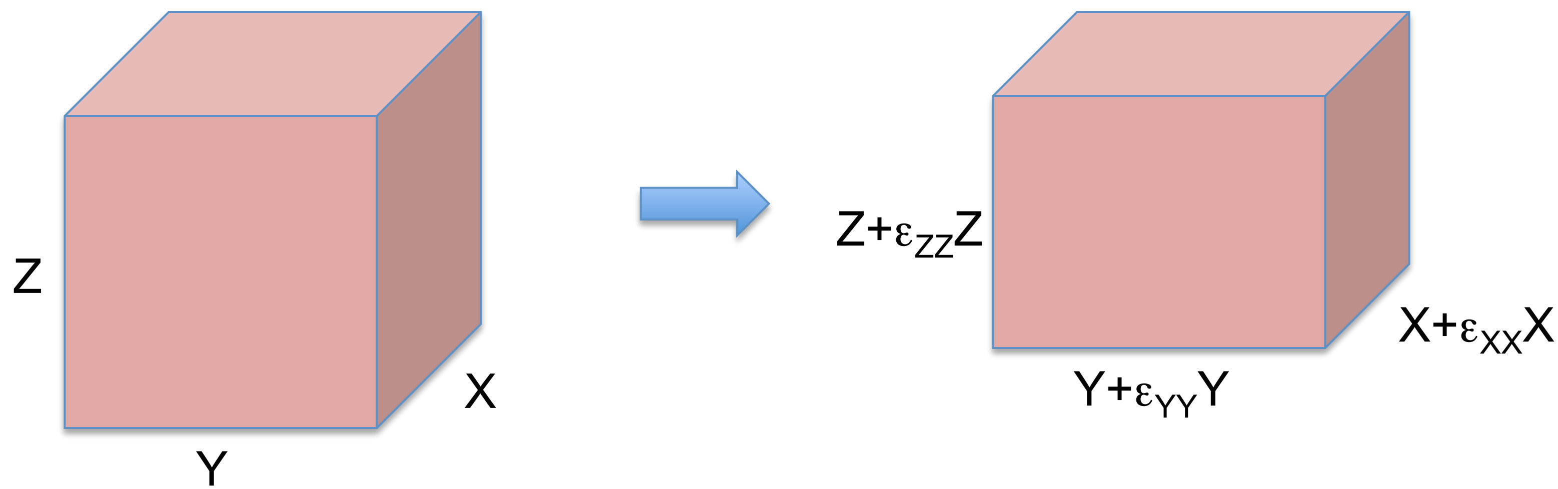
- Strains have direction. In the sketch there may also be a strain in the direction orthogonal to the line, but from this sketch we can't tell if it is deformation or rotation. If we assume no rotation,



- Define the ***shear strain*** to be the displacement in the orthogonal direction divided by the length, or strain ε_{shear} = (dy/L)= tanθ ~= θ
- There is a better definition of the shear strain, which we will come to in a little bit.

Dilatation

- Imagine a box that is compressed or extended equally in all dimensions (in the figure, the ε are negative):

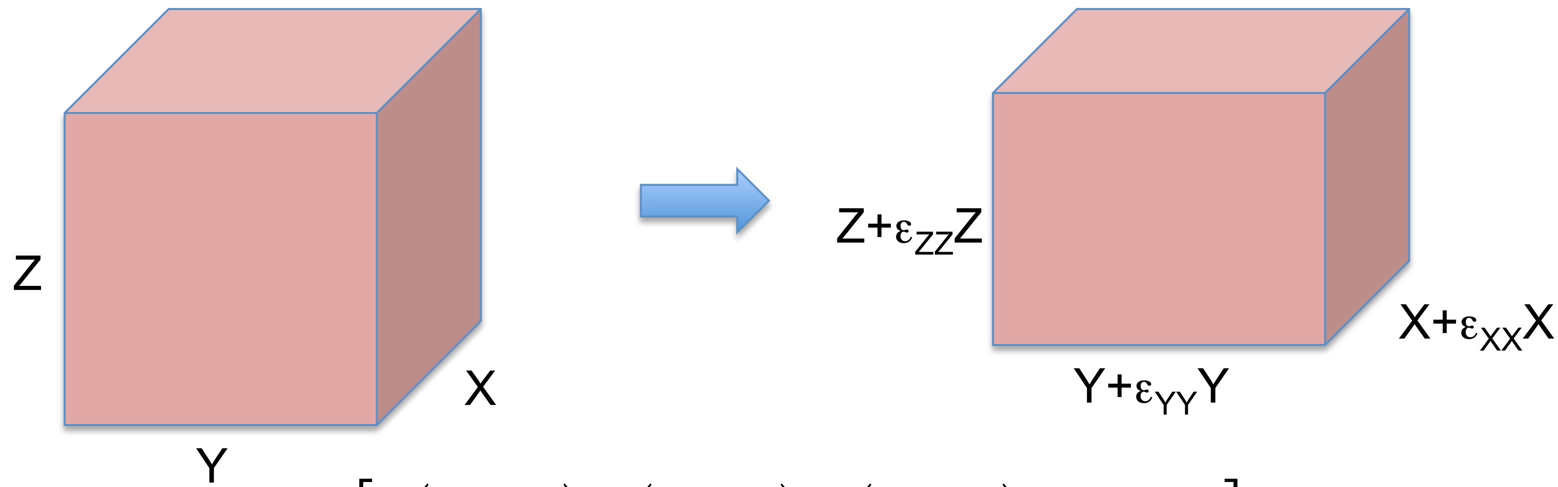


- The length changes (***axial strains***) in the x,y,z directions are
- X: $dL/L = \varepsilon_{XX}$
- Y: $dL/L = \varepsilon_{YY}$
- Z: $dL/L = \varepsilon_{ZZ}$

Note: positive = extension
Geologists often use positive = contraction

Dilatation

- Define the dilatation (Δ) as the fractional volume change



$$\Delta = \frac{\left[X\left(1+\varepsilon_{XX}\right)\cdot Y\left(1+\varepsilon_{YY}\right)\cdot Z\left(1+\varepsilon_{ZZ}\right)-X\cdot Y\cdot Z\right]}{X\cdot Y\cdot Z}$$

$$\Delta = \left[\left(1+\varepsilon_{XX}\right)\left(1+\varepsilon_{YY}\right)\left(1+\varepsilon_{ZZ}\right)-1\right]$$

$$\Delta = \left(1+\varepsilon_{XX}+\varepsilon_{YY}+\cancel{\varepsilon_{XX}\varepsilon_{YY}}\right)\left(1+\varepsilon_{ZZ}\right)-1$$

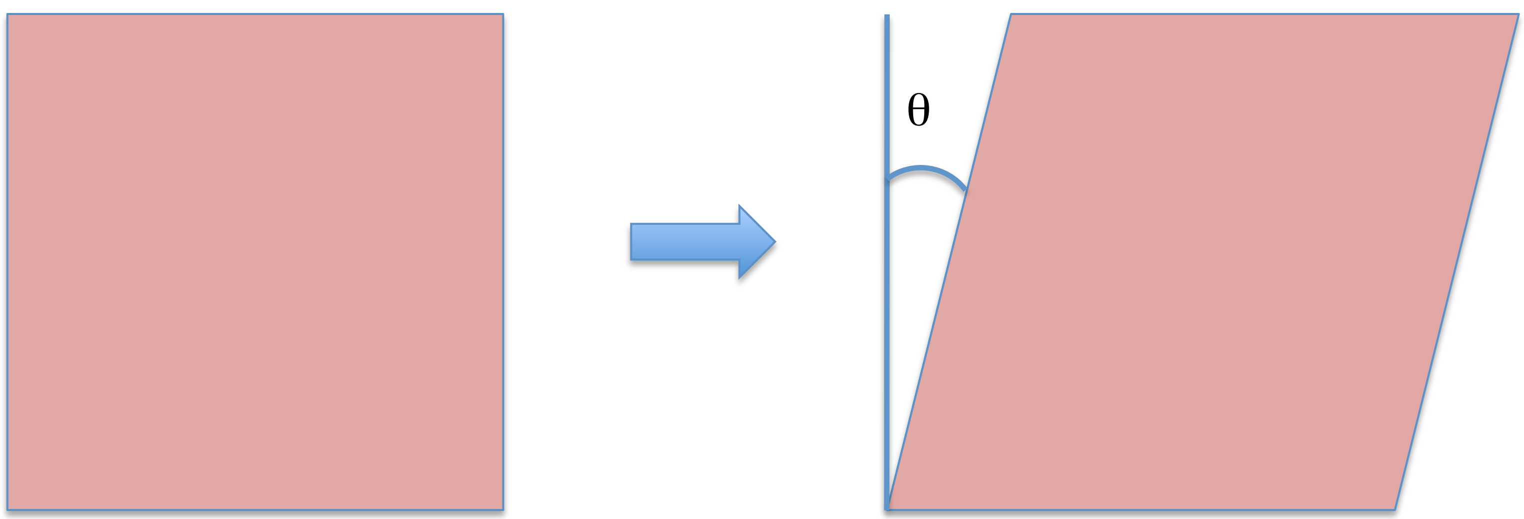
$$\Delta = 1+\varepsilon_{XX}+\varepsilon_{YY}+\varepsilon_{ZZ}-1$$

$$\Delta = \varepsilon_{XX}+\varepsilon_{YY}+\varepsilon_{ZZ}$$

Note: seismic P waves are traveling oscillations of Δ

Shear Strain

- Shear deformation is also very common. Define the shear strain in terms of the angle change of a side:

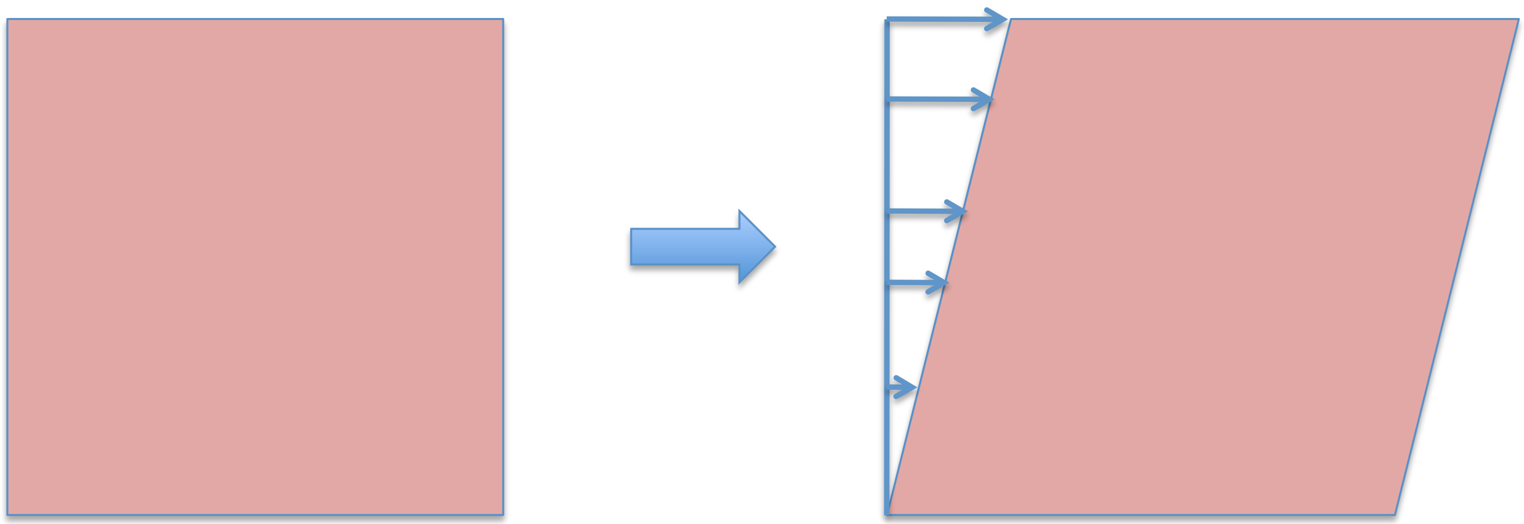


simple shear

- Shear strain = θ

Shear Strain

- Shear deformation is also very common. Define the shear strain in terms of the angle change of a side:

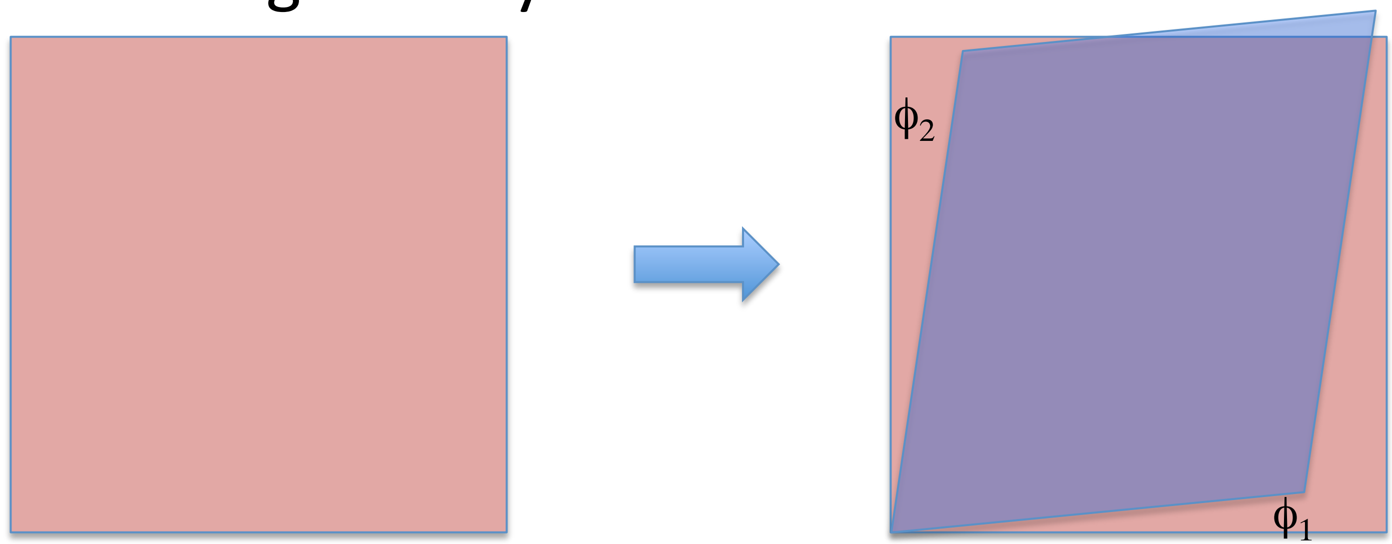


simple shear

- Shear strain = θ

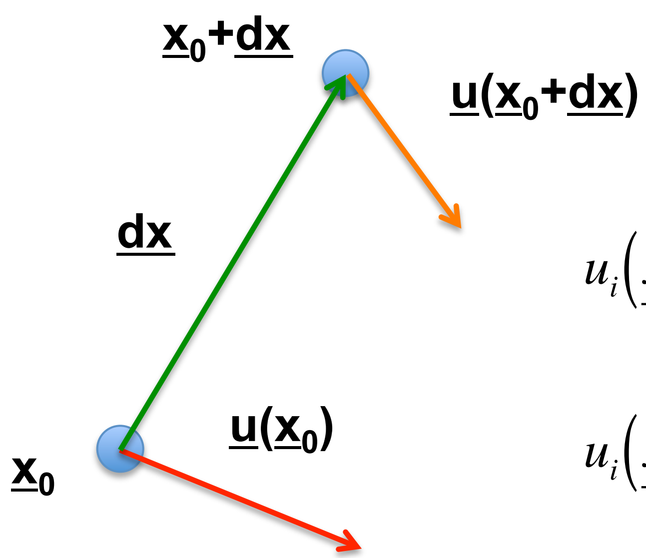
Shear Strain

- A bit more generally:



- Shear strain = $\phi_1 + \phi_2$
- If $\phi_1 = \phi_2$, then the strain is ***pure shear***
- Simple shear and pure shear differ only by a rigid rotation

Displacement



- Use a Taylor Series expansion to relate the two displacements:

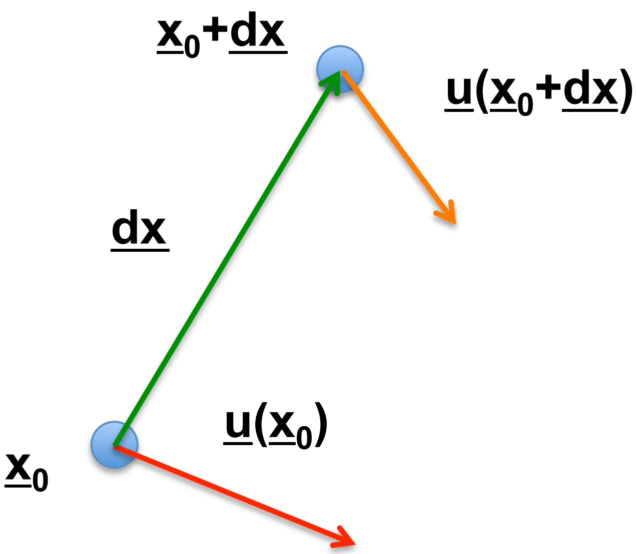
$$u_i\left(\underline{x}_0 + \underline{dx}\right) = u_i\left(\underline{x}_0\right) + \left(\frac{\partial u_i}{\partial x_j}\right)_{x=x_0} dx_j + \cdots$$

$$u_i\left(\underline{x}_0 + \underline{dx}\right) = u_i\left(\underline{x}_0\right) + \left(\frac{\partial u_i}{\partial x_1}\right) dx_1 + \left(\frac{\partial u_i}{\partial x_2}\right) dx_2 + \left(\frac{\partial u_i}{\partial x_3}\right) dx_3$$

This is a set of 3 equations, for i = 1, 2, 3
First term: translation
Remaining terms: rotation + strain
There are 9 values (du_i/dx_j) (i=1,3; j=1,3)

Displacement Gradient Tensor

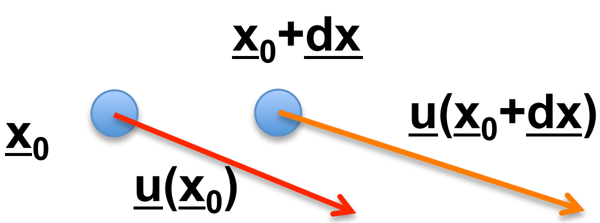
- These 9 values describe the deformation (strain) and rotation together:



$$\begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Extension in e_1 direction

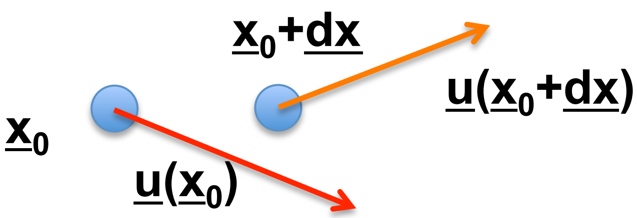
- These 9 values describe the deformation (strain) and rotation together:



$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \overset{\mathbf{0}}{\frac{\partial u_2}{\partial x_1}} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \overset{\mathbf{0}}{\frac{\partial u_3}{\partial x_1}} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Shear in e_2 direction

- These 9 values describe the deformation (strain) and rotation together:



$$\begin{bmatrix} \mathbf{0}\ \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \mathbf{0}\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Separating Rotation and Strain

- We can define this to be the sum of two tensors, a strain tensor and a rotation tensor. The rotation part will be anti-symmetric (remember the rotation matrix), and the strain part will be symmetric:

$$u_i\left(\underline{x_0} + \underline{dx}\right) = u_i\left(\underline{x_0}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)dx_j$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

Strain Tensor

Axial Strains

Shear Strains

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Rotation Tensor

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

In terms of our earlier coordinate rotation matrix:

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix} = \left(\beta_{ij}\right)\underline{x} - \underline{x}$$