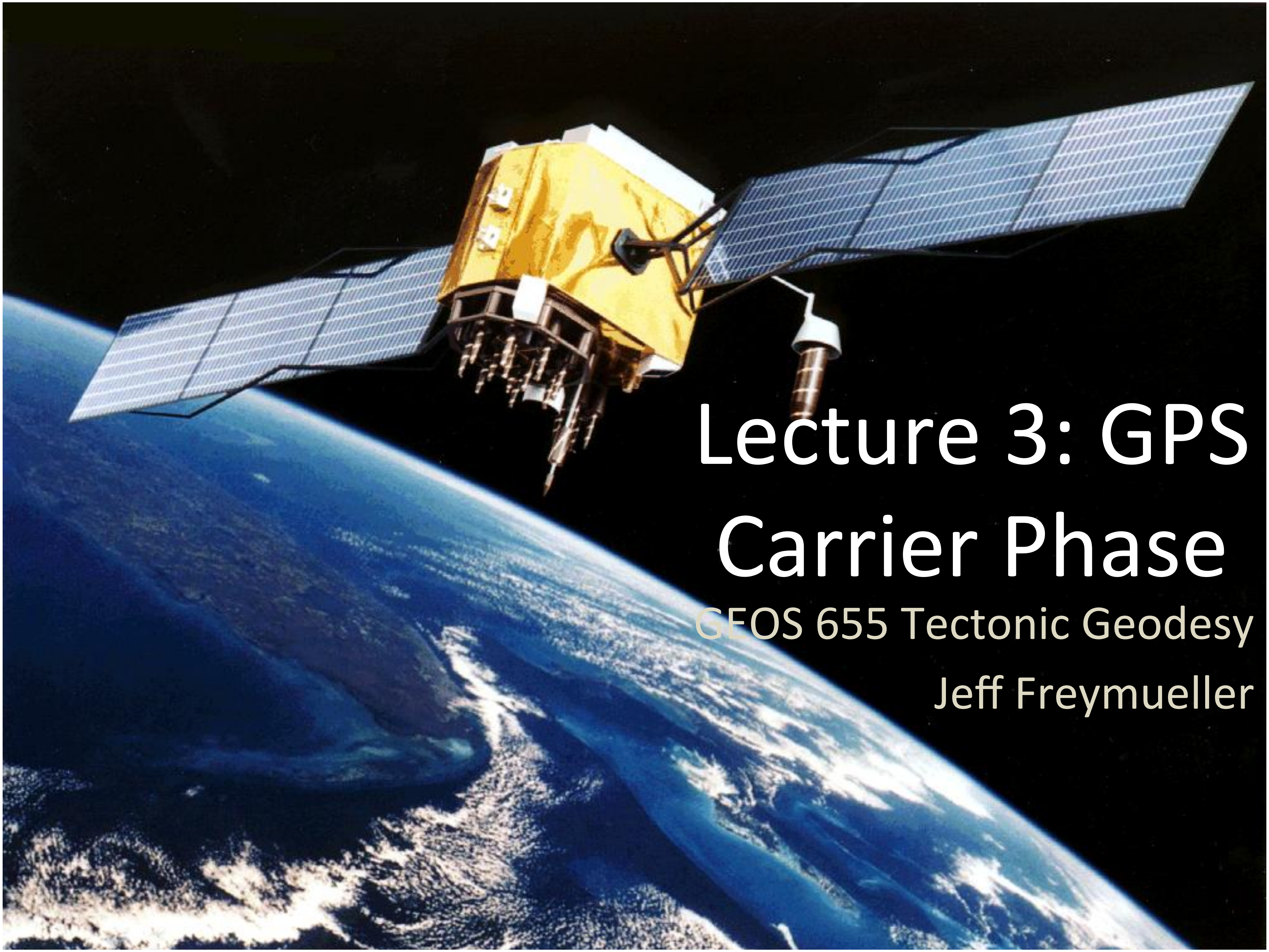


Lecture 3: GPS Carrier Phase

GEOS 655 Tectonic Geodesy

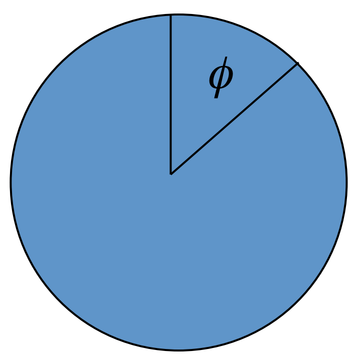
Jeff Freymueller

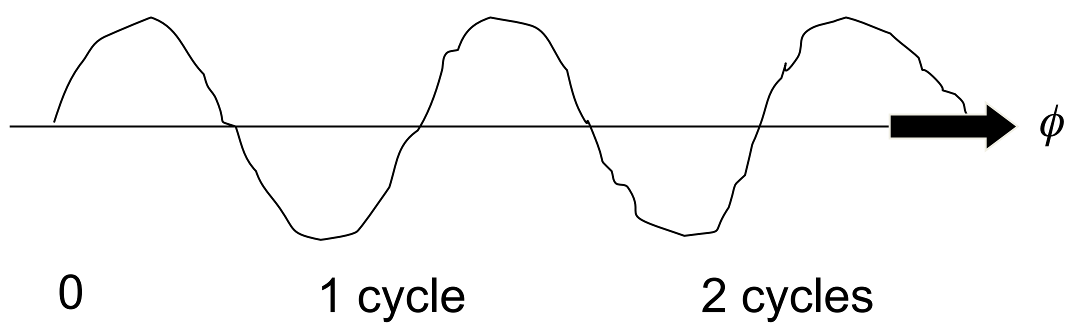
Thinking About Clocks

- All of our clocks are really periodic motion devices: rotation of earth, orbit of earth, pendulum, quartz oscillator, …
  - Therefore time is proportional to phase
  - $T(t) = k(\phi(t) - \phi_0)$
- Every clock keeps perfect time from its own perspective
  - Keep in our mind the distinction between "true" time and time kept by some clock (more important for the ultra-precise GPS phase measurements than for pseudorange).
- We can define the clock time in terms of the nominal frequency of the oscillator, f_0:
  - $T(t) = (\phi(t) - \phi_0)/f_0$

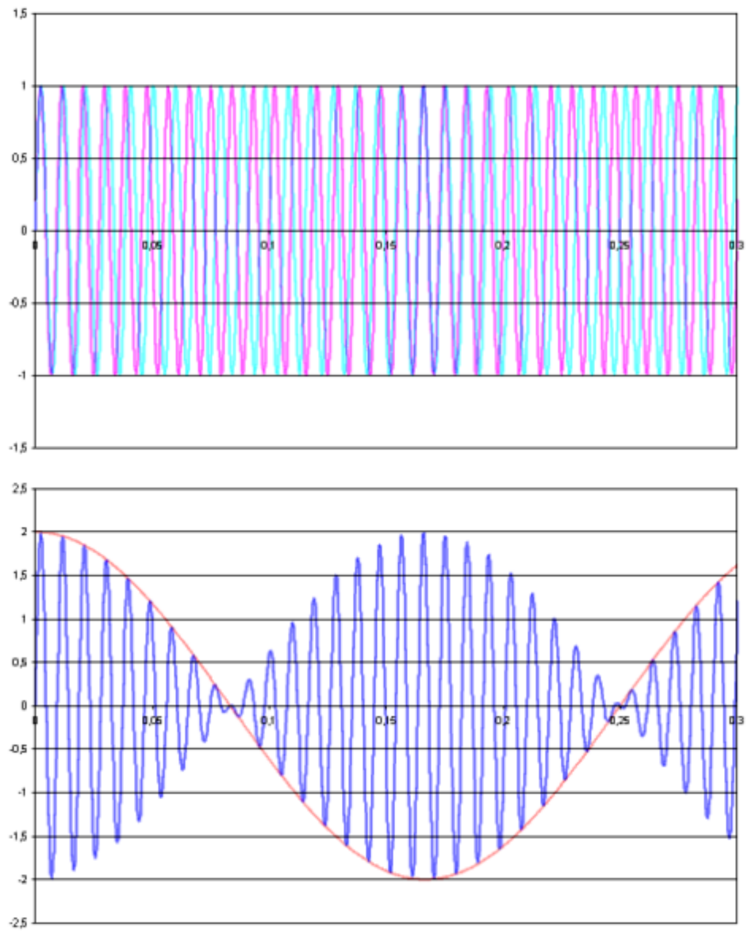
Phase Tracking

- Receiver measures changes in phase of carrier signal over time
  - First must remove codes to recover raw phase
  - Then track continuous phase, keeping record of the number of whole cycles
  - Phase has an integer ambiguity (initial value)
- Problems occur if receiver loses phase lock





Measurement Trick: Beat Phase



- Remove PRN code modulation by multiplying signal by code – removes phase shifts and recovers original carrier signal
- Mix received phase with reference phase signal
- Filter high frequency beats and measure phase of low frequency beat

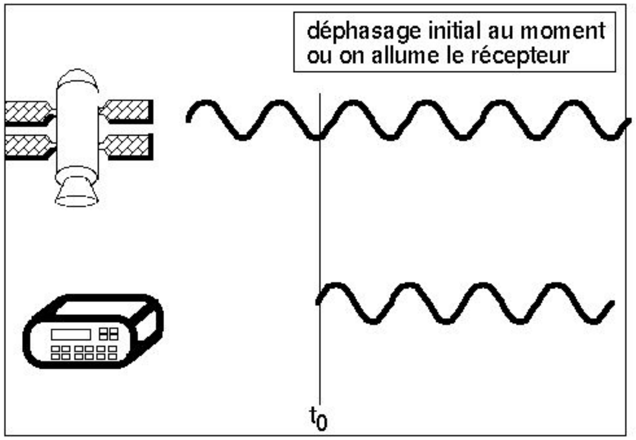
$$\sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2\cos\left(2\pi \frac{f_1 - f_2}{2} t\right) \sin\left(2\pi \frac{f_1 + f_2}{2} t\right)$$

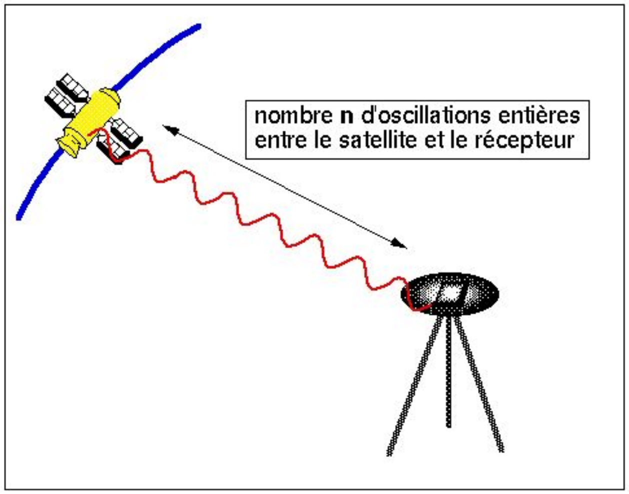
Measuring Beat Phase

- Doppler Shift shifts each SVs frequency slightly
- Receiver generates reference signal at nominal GPS frequency (f_G with phase ϕ_G)
- Beat phase ϕ_B and beat frequency f_B are
  - $\phi_B(t) = \phi_R(t) - \phi_G(t)$
  - $f_B = f_R - f_G$
- Beat frequency is much lower than nominal, easier to measure beat phase, but we can recover all variations in phase of the transmitted carrier signal from the beat phase

Phase Ambiguity

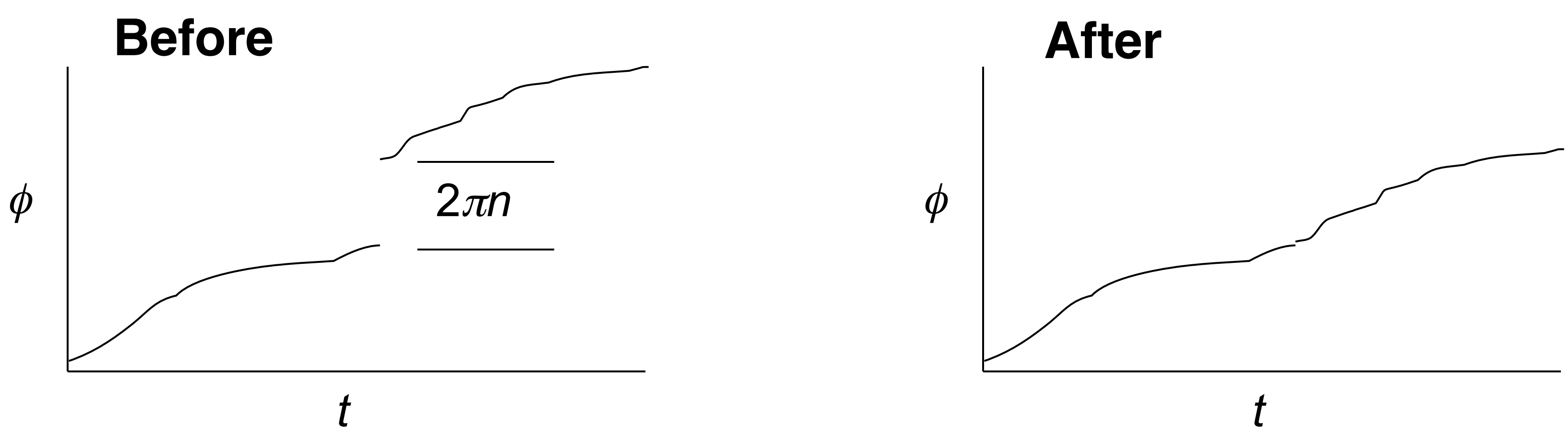
- One drawback of the phase is that we can add an arbitrary constant number of cycles to the transmitted carrier signal, and we would get exactly the same beat phase:
  - $\Phi + N = \phi_R - \phi_G$
  - Actual recorded phase is Φ
- Also, we must track the phase continuously. If we lose track of the phase over time, and start over, we get a different N.
  - Losing track is called a "cycle slip"





Loss of Lock (Cycle Slips)

- If receiver loses phase lock, there will be a jump of an integer number of cycles in the phase data
- This must be detected and repaired by the analysis software
- Slightly different procedures are usually applied multiple times to find all of the cycle slips



Carrier Beat Phase Model

- Observation of satellite S produces the phase observable Φ^S
  - $\Phi^S(T) = \phi(T) - \phi^S(T) - N^S$
  - Remember T is the receive time according to the receiver clock, and true time receive time t.
- We then use the fact that the signal at the receiver at receive time T is the same (same phase) as the signal at the transmitter at transmit time T^S
  - $\phi^S(x,y,z,T) = \phi^S(x^S,y^S,z^S,T^S)$
- Like with pseudorange, we'll eventually have to deal with how T relates to T^S

Carrier Beat Phase Model 2

- We can use the clock-time equivalency, $T(t) = (\phi(t) - \phi_0)/f_0$, to write the observable in terms of times:
  - $\Phi^S(T) = f_0 T + \phi_0 - f_0 T^S - \phi_0^S - N^S$
  - $\Phi^S(T) = f_0(T - T^S) + \phi_0 - \phi_0^S - N^S$
- The last three terms are all arbitrary constants and can be combined into a single bias term
- We'll put this in terms of distance (range) instead of phase, and add subscripts to identify each station
  - $L_A^j(T_A) = \lambda_0 \Phi^S(T_A) = \lambda_0 f_0(T_A - T^j) + \lambda_0 (\phi_{0A} - \phi_0^j - N_a^j)$
  - Carrier phase bias $B_A^j = \lambda_0 (\phi_{0A} - \phi_0^j - N_a^j)$
  - Also note that $\lambda_0 f_0 = c$

Observation Equations

- Compare the phase observation model with the pseudorange model:
  - $L_A^j(T_A) = c(T_A - T^j) + B_a^j$
  - $P_A^j(T_A) = c(T_A - T^j)$
  - Exactly the same except for the phase bias!
- We need to add more terms to deal with the clock errors and with path delay terms
  - $L_A^j(T_A) = \rho_A^j(t_A, t^j) + c\tau_A - c\tau^j + Z_a^j - I_A^j + B_A^j$
  - $P_A^j(T_A) = \rho_A^j(t_A, t^j) + c\tau_A - c\tau^j + Z_a^j + I_A^j$
  - Path delay terms are Z for the troposphere, and I for ionosphere. We'll come back to these later on.

Time Tag Bias and Light Time Equation

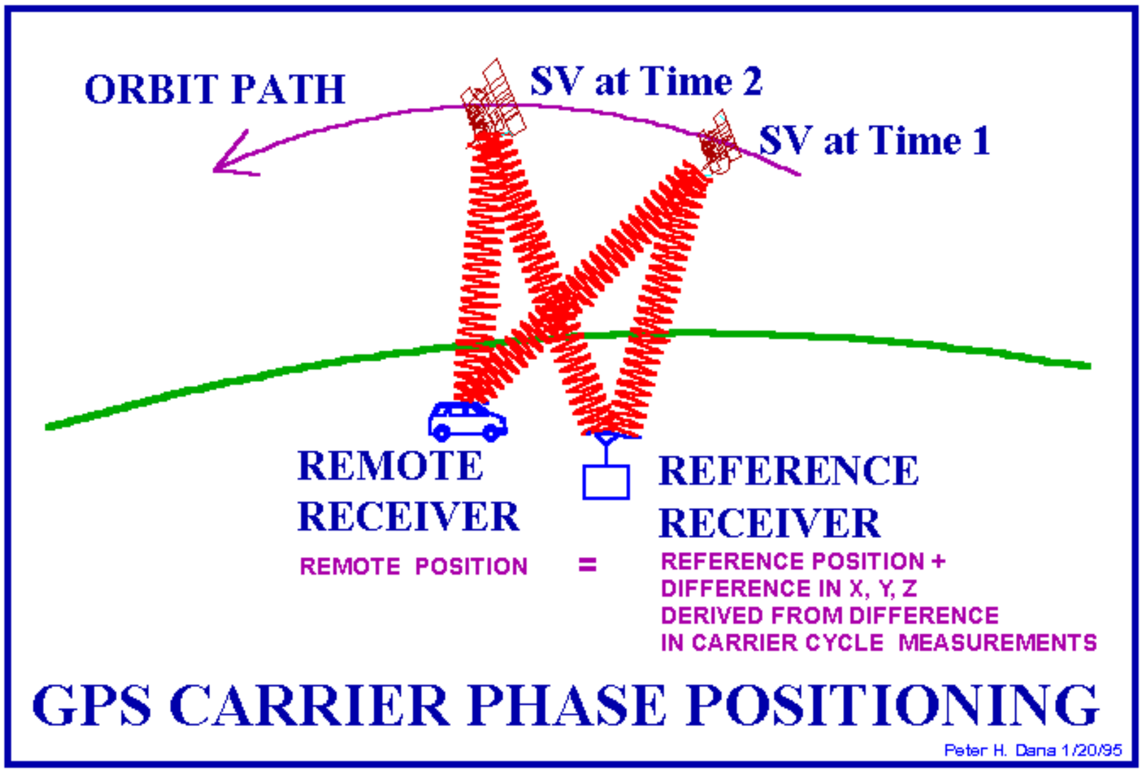
- We still need to solve the Light Time Equation to compute geometric range $\rho_A^j(t_A, t^j)$ correctly.
- When we dealt with this in the pseudorange equations, I said we could ignore the difference between the true receive time t_A and the receiver clock's receive time T_A
  - $t_A = T_A - \tau_A$
- To go to 1 mm precision, we need to use an estimate of τ_A accurate to 1 microsecond. That is more demanding than GPS receiver clocks can actually be synchronized.
  - This is one reason for the temp. stabilized clocks in the old TI-4100 receiver

Several Solutions to This Problem

Any of these would work: in general software usually uses #1 or #3

1. Estimate the receiver clock bias first using a pseudorange solution
2. Iterate the least squares solution using both phase and pseudorange data
   - Start out with receiver clock bias = 0, estimate
   - Repeat estimation assuming last clock bias
3. Use an estimate of true transmit time given satellite clock error and approximate station position
   - Requires station position to ~300 m
4. Expand range model in a Taylor series, adding range-rate term

Differencing Techniques

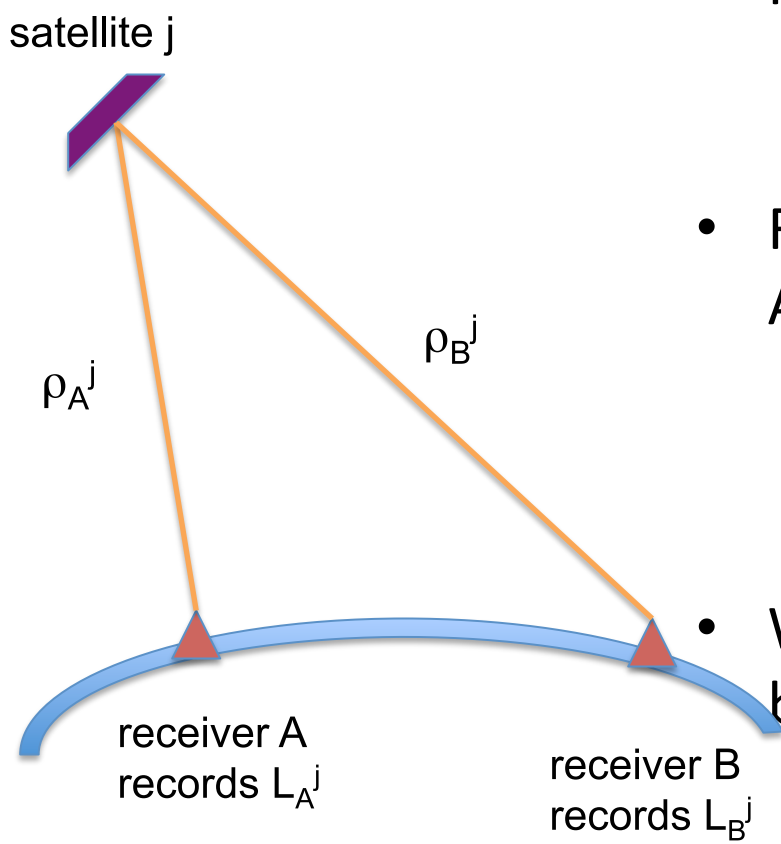


- Receiver and satellite clock biases can be removed by differencing data from multiple satellites and/or receivers.
  - Difference between receivers ("single difference") removes satellite clock
  - Difference between satellites ("single difference") removes receiver clock
  - Difference of differences ("double difference") removes both clocks

Advantages/Disadvantages of Differencing

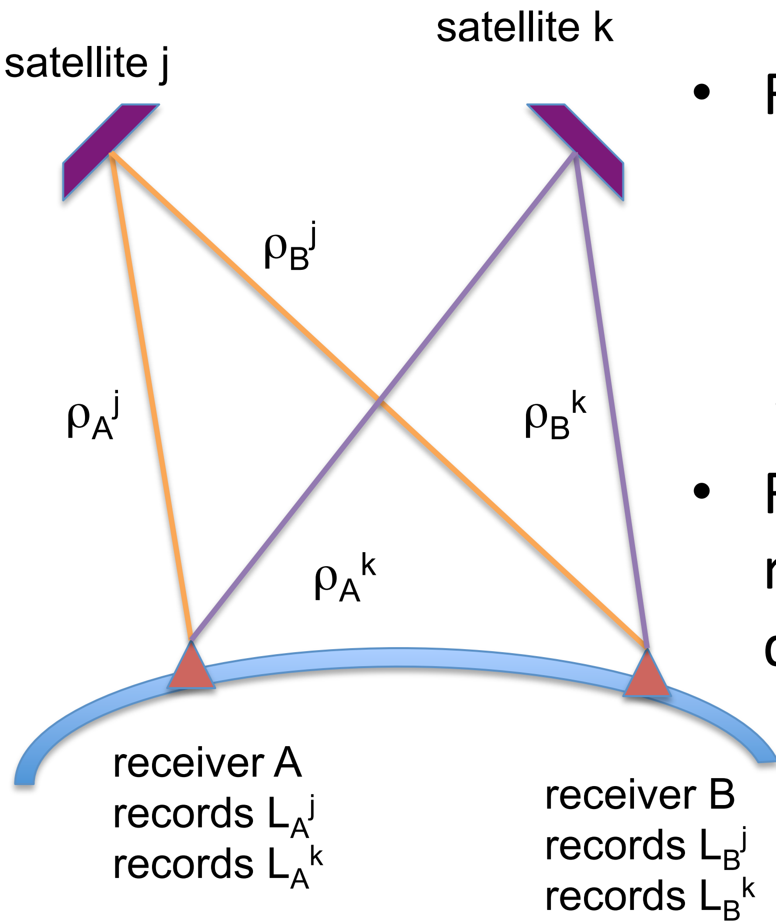
- Advantage
  - Removes clock errors, which are a pain
  - Makes for faster estimation (fewer parameters if you do not need to estimate clock error at every observation time)
  - Phase bias parameters reduce to integer values
- Disadvantages
  - Requires a method to select differences
  - Requires additional bookkeeping
  - Notation gets messy

Single Difference



- Receivers A and B observe:
  - $L_A^j = \rho_A^j + c\tau_A - c\tau^j + B_A^j$
  - $L_B^j = \rho_B^j + c\tau_B - c\tau^j + B_B^j$
- Form a difference between receivers A and B
  - $\Delta L_{AB}^j = L_A^j - L_B^j$
  - $\Delta L_{AB}^j = (\rho_A^j - \rho_B^j) + (c\tau_A - c\tau_B) + (B_A^j - B_B^j)$
  - $\Delta L_{AB}^j = \Delta\rho_{AB}^j + c\Delta\tau_{AB} + \Delta B_{AB}^j$
- We use Δ to indicate a difference between ground receivers.

Double Difference

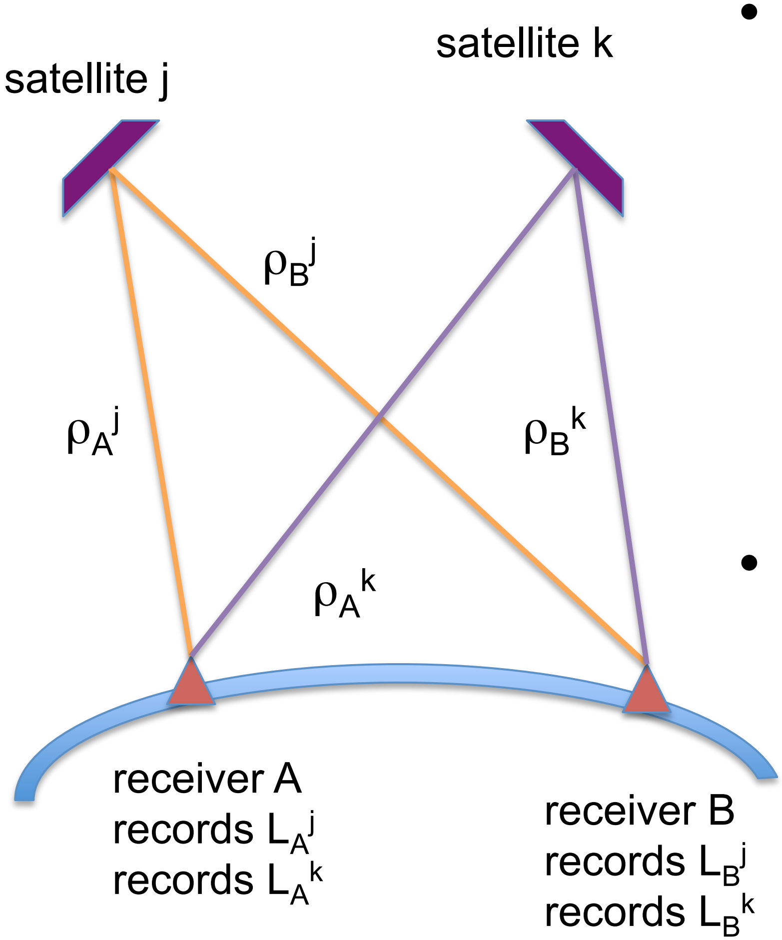


- Receivers A and B observe:
  - $L_A^j = \rho_A^j + c\tau_A - c\tau^j + B_A^j$
  - $L_A^k = \rho_A^k + c\tau_A - c\tau^k + B_A^k$
  - $L_B^j = \rho_B^j + c\tau_B - c\tau^j + B_B^j$
  - $L_B^k = \rho_B^k + c\tau_B - c\tau^k + B_B^k$
- Form the single difference between receivers A and B, and then difference between satellites j and k:
  - $\Delta\Delta L_{AB}^{jk} = \Delta L_{AB}^j - \Delta L_{AB}^k$
  - $\Delta\Delta L_{AB}^{jk} = (\Delta\rho_{AB}^j - \Delta\rho_{AB}^k) + (\Delta B_{AB}^j - \Delta B_{AB}^k)$
  - $\Delta\Delta L_{AB}^{jk} = \Delta\Delta\rho_{AB}^{jk} + \Delta\Delta B_{AB}^{jk}$
- We use $\Delta\Delta$ to indicate a double difference.

Double-Differenced Ambiguity

- The double-differenced phase ambiguities become exactly integers:
  - $\Delta\Delta B_{AB}^{jk} = \Delta B_{AB}^{j} - \Delta B_{AB}^{k} = \lambda_0 \Delta N_{AB}^{jk}$
- Each B has three parts:
  - $B_A^j = \lambda_0 (N_A^j + \phi_{0A} - \phi_0^j)$
  - The receiver bias ϕ_{0A} is common to all satellites, and differences out like the receiver clock.
  - The satellite bias ϕ_0^j is common to all receivers, and differences out like the satellite clock.
- There are some clever techniques to remove the phase ambiguity completely, if you can resolve it to the correct integer.

Triple Difference



- The triple difference adds a difference in time. If you difference the double-differenced observations from one epoch in time to those of the previous epoch, you get a triple difference:
  - $\Delta\Delta\Delta L_{AB}^{jk}(i+l, i) = \Delta\Delta\Delta\rho_{AB}^{jk}(i+l, i)$
- We use $\Delta\Delta$ to indicate a double difference. The triple difference also removes most of the geometric strength from GPS, so it produces only weakly determined positions. But the triple difference canbe applied to kinematic problems.

Final Notes on Differencing

- When you difference between receivers, then in effect you are now estimating the baseline vector between the two receivers, rather than the two positions.
- You have to take some care in choosing which differences to use
  - Cannot use linearly dependent observations
  - Must be careful in choosing baselines to difference, satellites to difference between.
- Each software does it differently
- Some softwares do not difference at all, but estimate clock errors instead.

Design Matrix for Phase

- Let's look at one row of the design matrix for a case of 2 stations, 4 satellites. We will consider station A to be fixed, and estimate station B.
  - 3 double differenced observations
  - 3 double difference ambiguities
  - 6 parameters! Need ≥ 2 epochs of data!

$$A_{AB}^{(24)}(i) = \begin{pmatrix} \frac{\partial L_{AB}^{(24)}(i)}{\partial x_B} & \frac{\partial L_{AB}^{(24)}(i)}{\partial y_B} & \frac{\partial L_{AB}^{(24)}(i)}{\partial z_B} & \frac{\partial L_{AB}^{(24)}(i)}{\partial N_{AB}^{21}} & \frac{\partial L_{AB}^{(24)}(i)}{\partial N_{AB}^{23}} & \frac{\partial L_{AB}^{(24)}(i)}{\partial N_{AB}^{24}} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ N_{AB}^{21} \\ N_{AB}^{23} \\ N_{AB}^{24} \end{pmatrix}$$

$$A_{AB}^{(24)}(i) = \begin{pmatrix} \frac{\partial \rho_{AB}^{(24)}(i)}{\partial x_B} & \frac{\partial \rho_{AB}^{(24)}(i)}{\partial y_B} & \frac{\partial \rho_{AB}^{(24)}(i)}{\partial z_B} & 0 & 0 & -\lambda_0 \end{pmatrix}$$

Design Matrix for Phase 2

- The phase design matrix is the same as the pseudorange version, except for the phase ambiguities. The double-differenced version in the last slide resolves to:

$$\frac{\partial \rho_{AB}^{(24)}(i)}{\partial x_B} = \frac{\partial}{\partial x_B}\left(\rho_A^{(2)}(i) - \rho_B^{(2)}(i) - \rho_A^{(4)}(i) + \rho_B^{(4)}(i)\right)$$

$$\frac{\partial \rho_{AB}^{(24)}(i)}{\partial x_B} = \frac{\partial \rho_B^{(4)}(i)}{\partial x_B} - \frac{\partial \rho_B^{(2)}(i)}{\partial x_B}$$

$$\frac{\partial \rho_{AB}^{(24)}(i)}{\partial x_B} = \frac{x_{B0} - x^{(4)}(i)}{\rho_B^{(4)}(i)} - \frac{x_{B0} - x^{(2)}(i)}{\rho_B^{(2)}(i)}$$

- If we compare the double-differenced version to the undifferenced version:

$$\begin{matrix} (undifferenced) & & (double-differenced) \\ \dfrac{x_{B0} - x^{(4)}(i)}{\rho_B^{(4)}(i)} & \Rightarrow & \dfrac{x_{B0} - x^{(4)}(i)}{\rho_B^{(4)}(i)} - \dfrac{x_{B0} - x^{(2)}(i)}{\rho_B^{(2)}(i)} \end{matrix}$$

Weighted Least Squares

- To properly estimate the position with differencing, you have to account for the fact that the measurements are now correlated
  - Because the observation from station A, satellite j may be used in more than one differenced observation.
- The first step is to define a weight matrix. The weight matrix must be symmetric ($w_{ji} = w_{ij}$). The matrix at right shows a weight matrix for equally-weighted and uncorrelated data

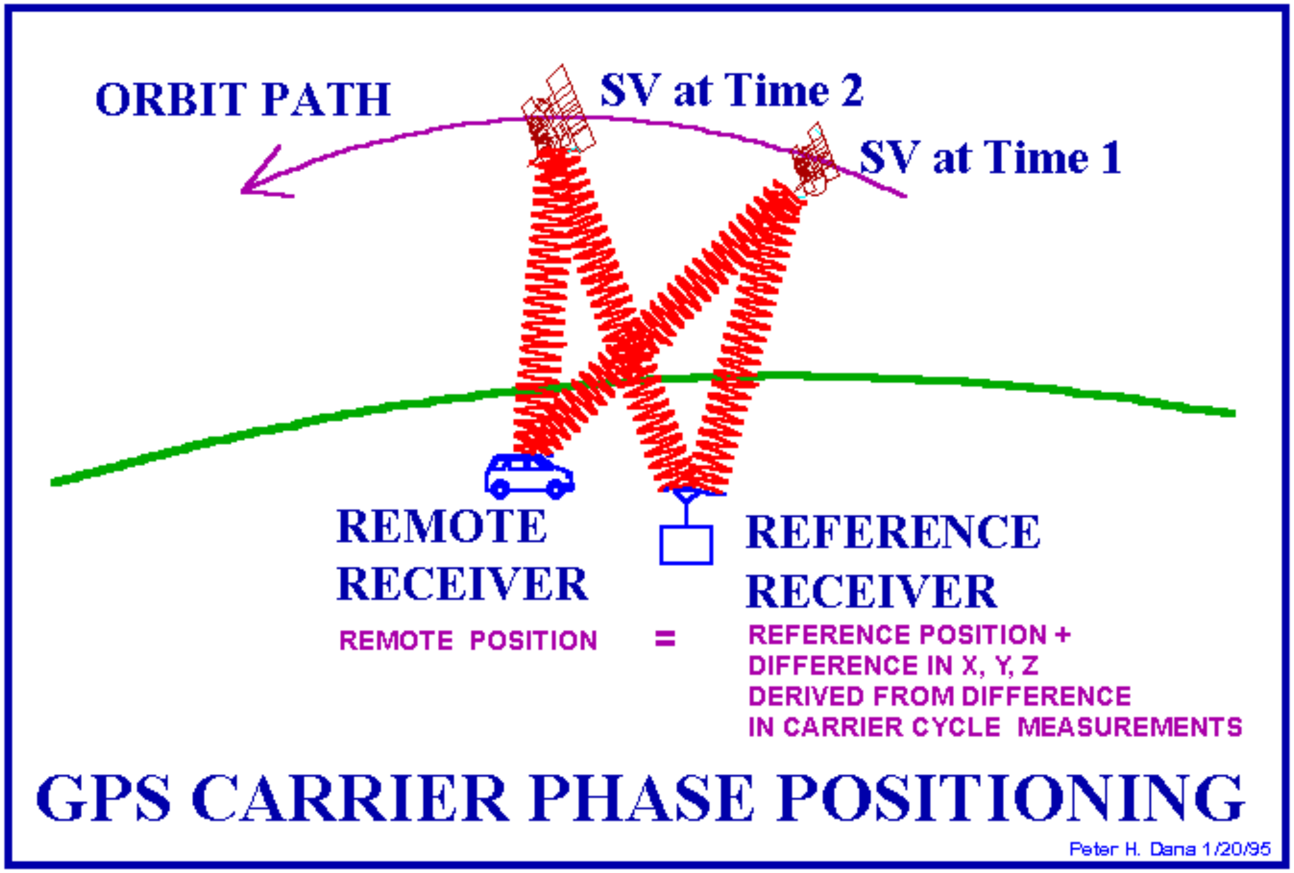
$$W = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix}$$

$$W = \sigma^{-2} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Weighted Least Squares Solution

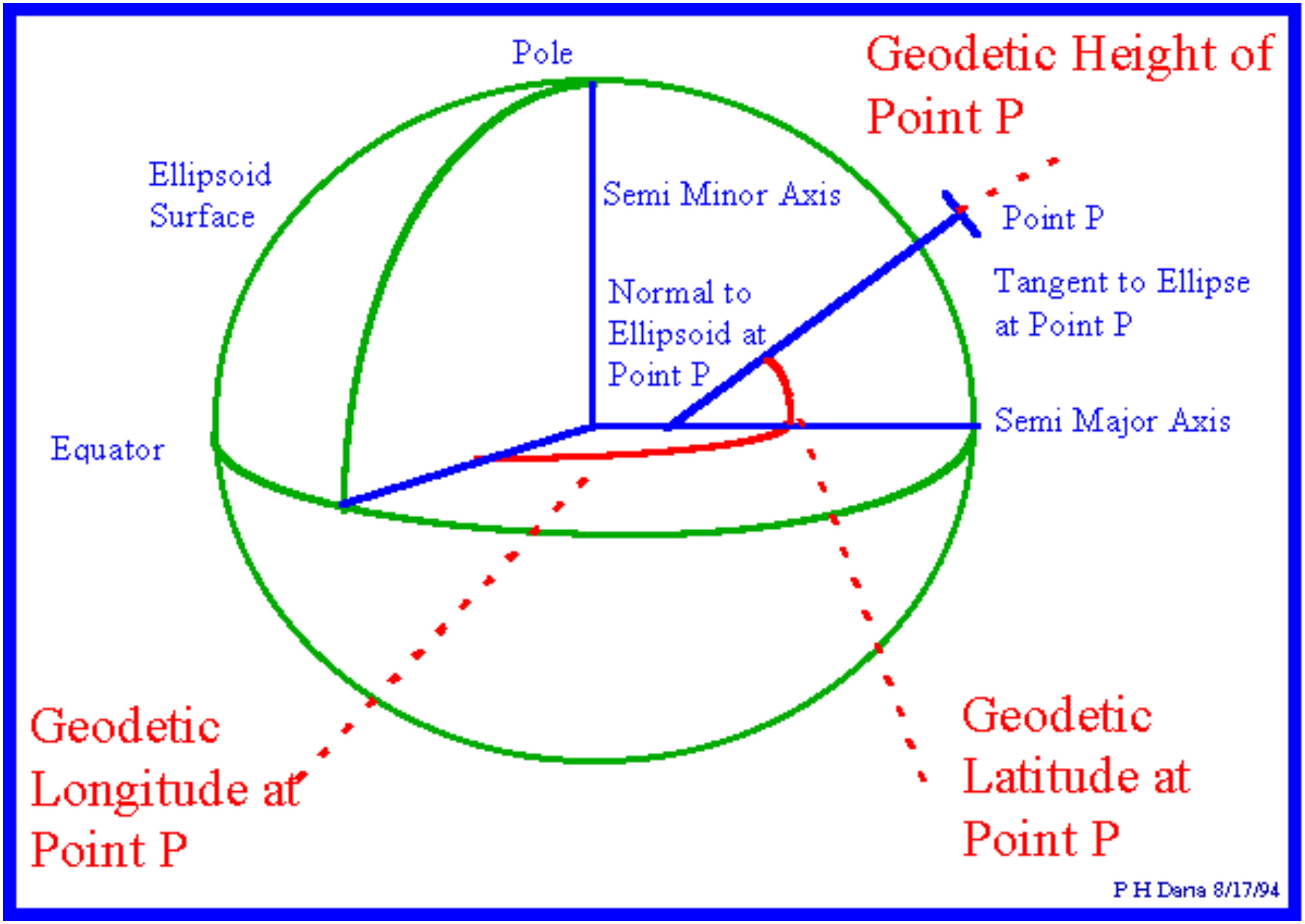
- The weight matrix W is going to be the inverse of the data covariance matrix $W = C^{-1}$
- The weighted least squares solution looks very similar to the unweighted version
  - Unweighted: $x' = (A^TA)^{-1}A^Tb$
  - Weighted: $x' = (A^TWA)^{-1}A^TWb$
- The model covariance is also very similar
  - Unweighted: $C_x = (A^TA)^{-1}$
  - Weighted: $C_x = (A^TWA)^{-1}$

Product is Differential Position



Or a set of relative positions (all sites in network relative to each other)

XYZ to LLH



Pole
Geodetic Height of Point P
Ellipsoid Surface
Semi Minor Axis
Point P
Tangent to Ellipse at Point P
Normal to Ellipsoid at Point P
Semi Major Axis
Equator
Geodetic Longitude at Point P
Geodetic Latitude at Point P
P H Dana 8/17/94

Ionospheric Calibration

- To a very good approximation, the path delay due to ionospheric refraction is proportional to $1/f^2$
- The phase is ***advanced***, while the pseudorange data are ***delayed***
  - Information travels at group velocity
- Specifically, the path delay is $(40.3/f^2)$TEC, where TEC is the total electron content. This path delay can be as large as meters.
  - The delay term $I_a^j = 40.3TEC/f^2$; $f = f_1$ for L1, f_2 for L2

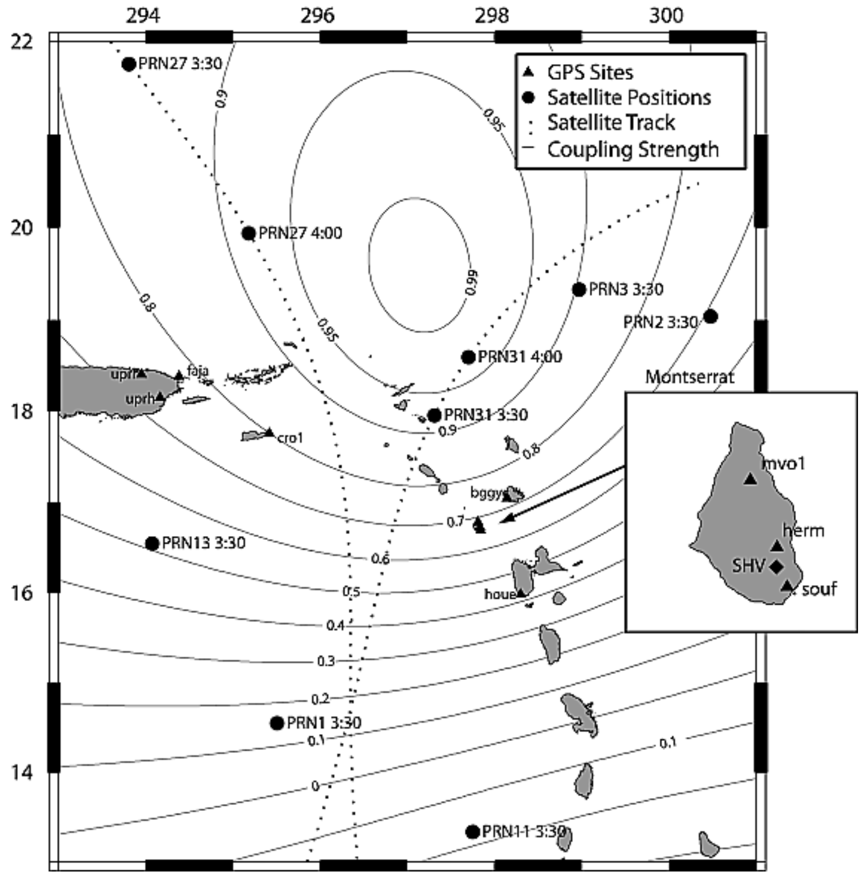
Ionosphere-free Combination

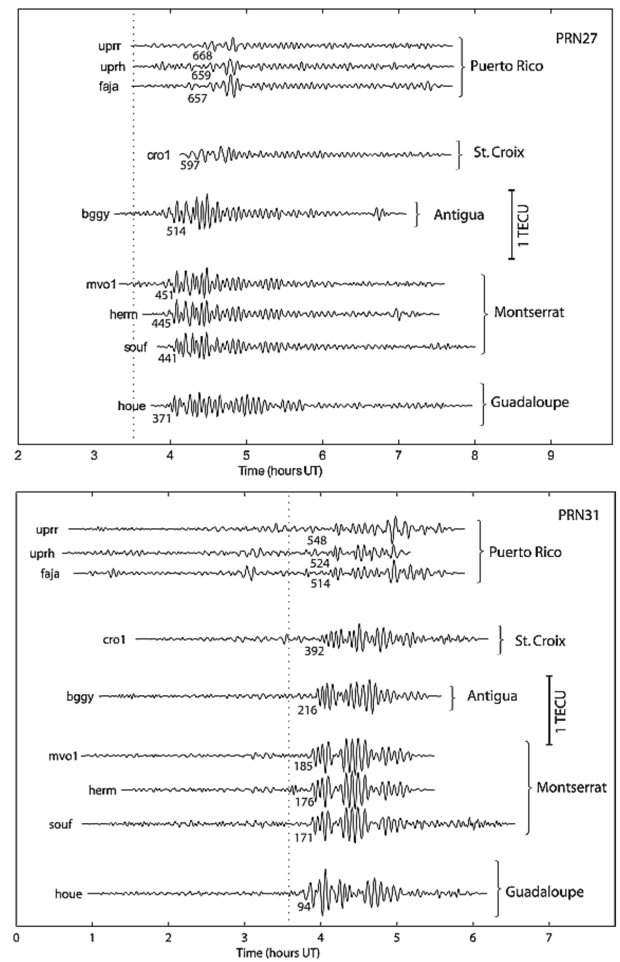
- We can remove the effects of the ionosphere by forming a linear combination of the data at the two frequencies
  - $L_C = -f_1^2/(f_2^2 - f_1^2)L_1 + f_2^2/(f_2^2 - f_1^2)L_2$
  - $P_C = -f_1^2/(f_2^2 - f_1^2)P_1 + f_2^2/(f_2^2 - f_1^2)P_2$
- Try it: For L1 and L2, the biases are $(40.3TEC/f_1^2, 40.3TEC/f_2^2)$
- Note that the two coefficients sum to 1. They have values of approximately (-1.54, 2.54)
- This removes all ionospheric effects except for a $1/f^4$ dependence. There are now "second-order ionosphere" models coming into use, which have an impact on positions at the few mm level or less.

Ionosphere-Only Combination

- If one linear combination of the data can remove the ionosphere, then there must be an orthogonal combination that has nothing but the ionosphere:
- $L1 - L2 = -40.3TEC(1/f_1^2 - 1/f_2^2) + B_1 - B_2$
- $P1 - P2 = -40.3TEC(1/f_1^2 - 1/f_2^2)$
  - Here B1 and B2 are the phase ambiguities
  - All other terms cancel: $L_A^j(T_A) = \rho_A^j(T_A, t^j) + c\tau_A - c\tau^j + T_A^j - I_A^j + B_A^j$
  - This combination may be used for ionospheric studies, but also to study waves propagating in ionopshere from earthquakes or volcanic explosions.

Ionospheric Waves from an Explosion at Montserrat

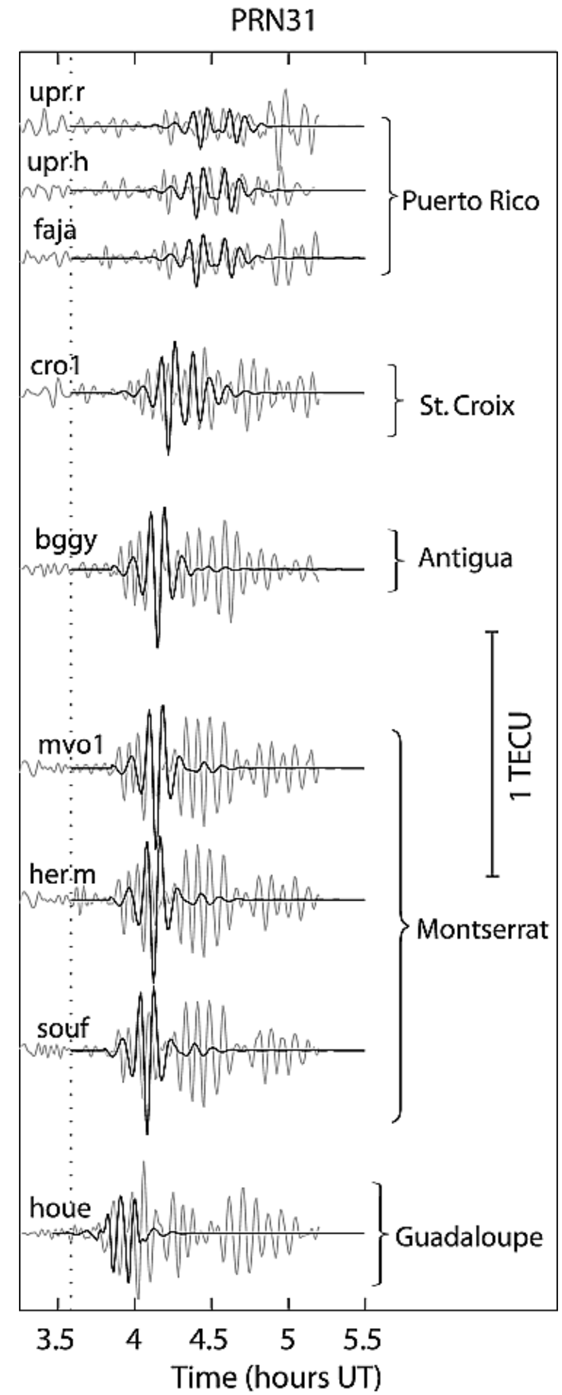




Dautermann et al., 2009, JGR (doi:10.1029/2008JB005722)

Physical Mechanism

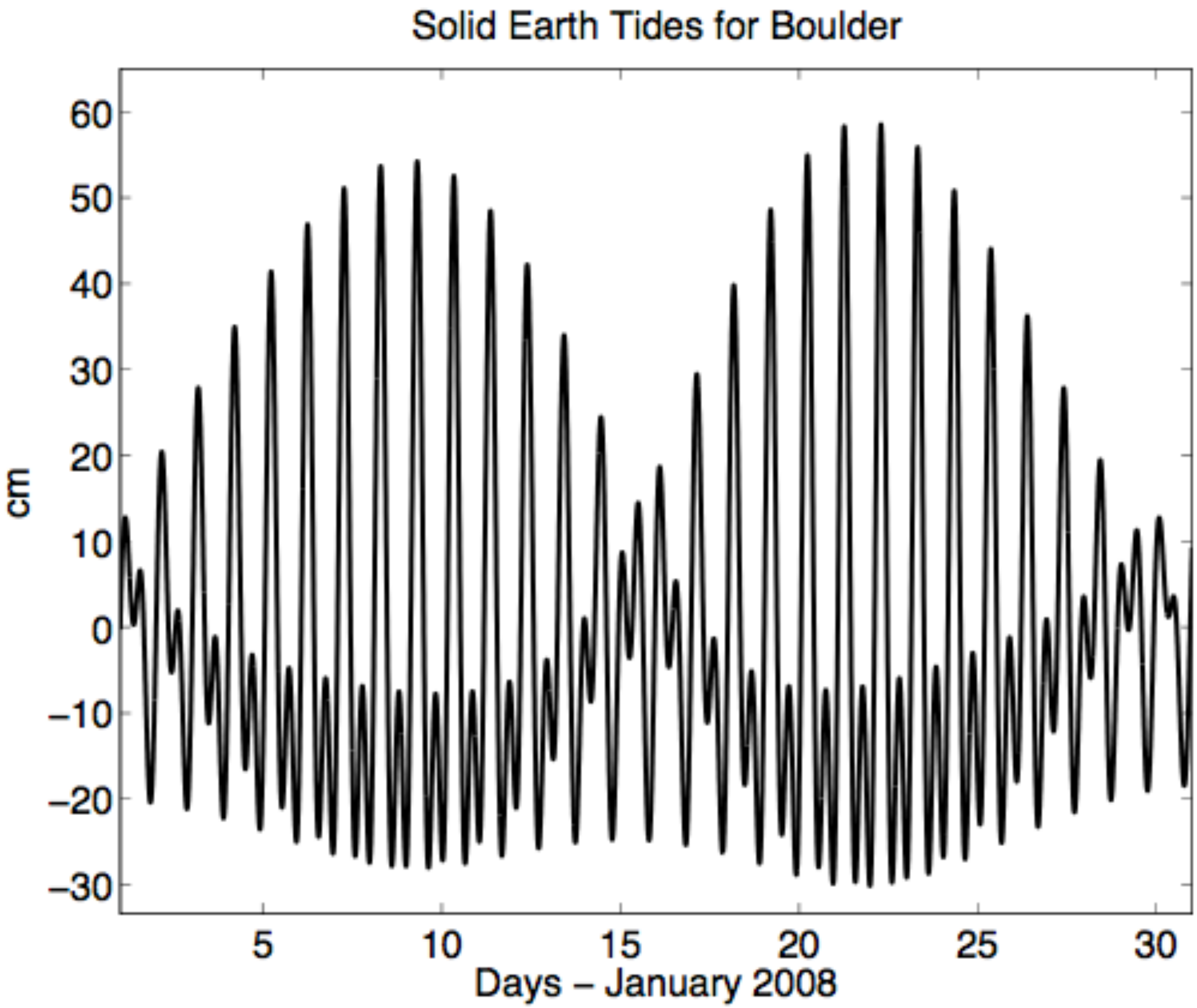
- Volcanic explosions or shallow earthquakes trigger acoustic and gravity waves that propagate in the atmosphere at infrasonic speeds.
- At ionospheric heights, coupling between neutral particles and free electrons induces variations of electron density detectable with dual-frequency Global Positioning System (GPS) measurements.
- Ray tracing of neutral atmo pressure wave
- Total acoustic energy release of 1.53×10^{10} J for the primary explosion event at Soufrière Hills Volcano associated with the peak dome collapse.
- This method can be applied to any explosion of sufficient magnitude, provided GPS data are available at near to medium range from the source.



Some Other Important Models

- Tropospheric delay (estimated)
  - Hopfield, Saastamoinen, Lanyi, Niell, **GMF, VMF1**
- Earth tides (well known)
  - Up to ~70 cm peak to peak
- Ocean Tidal Loading
  - Response of solid earth to changing load of ocean tides
- Antenna Phase Center variations with elevation
  - Phase center is the point on the antenna that we actually measure distances to
  - It is an imaginary point in space, not a physical point

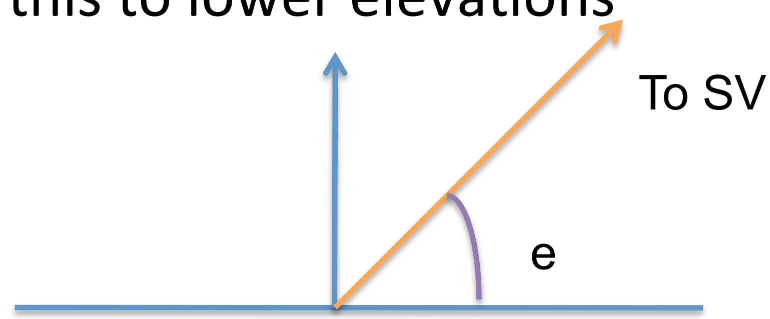
Vertical Solid Earth Tide - over one month



K. Larson, University of Colorado

Troposphere

- Tropospheric path delay affects both frequencies identically. It has two components.
  - "Dry" delay: due to air mass (~ proportional to pressure)
  - "Wet" delay: due to integrated water vapor along path
- Delay in both cases is largest at low elevations above the horizon, because the path length through atmosphere is longer there.
- In practice, we estimate a zenith delay, and use a "mapping function" of elevation angle to map this to lower elevations
  - Mapping function is ~1/sin(e)

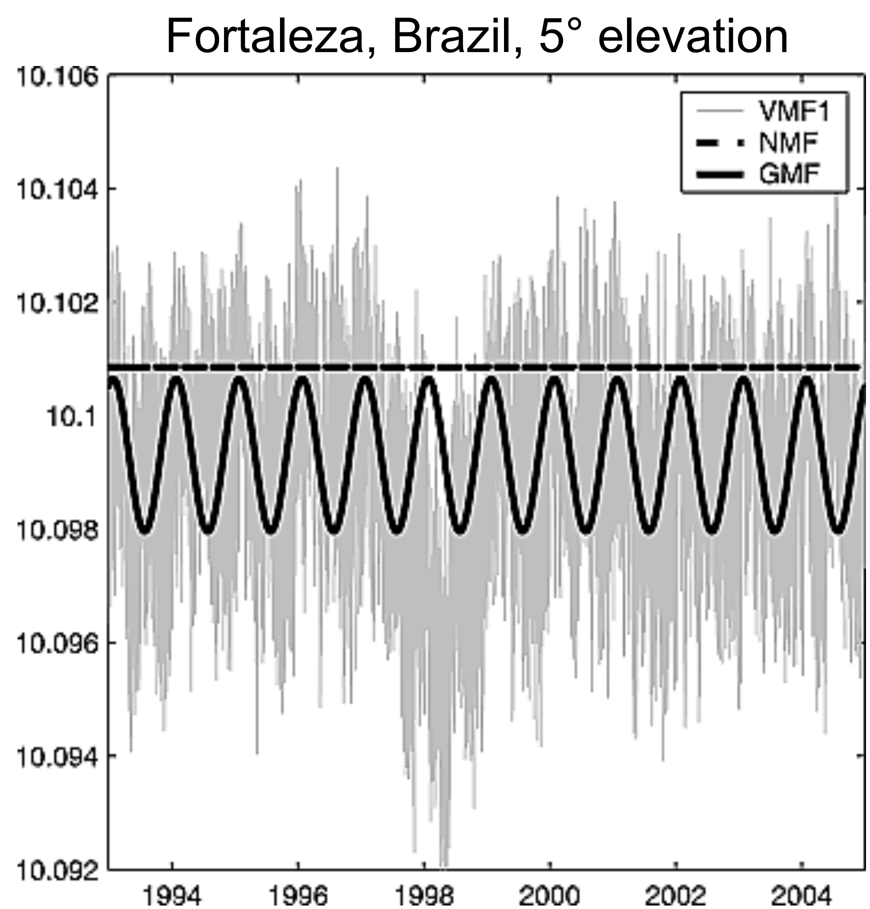


Wet Tropospheric Mapping Function

- Detailed form of mapping function is a continued fraction:

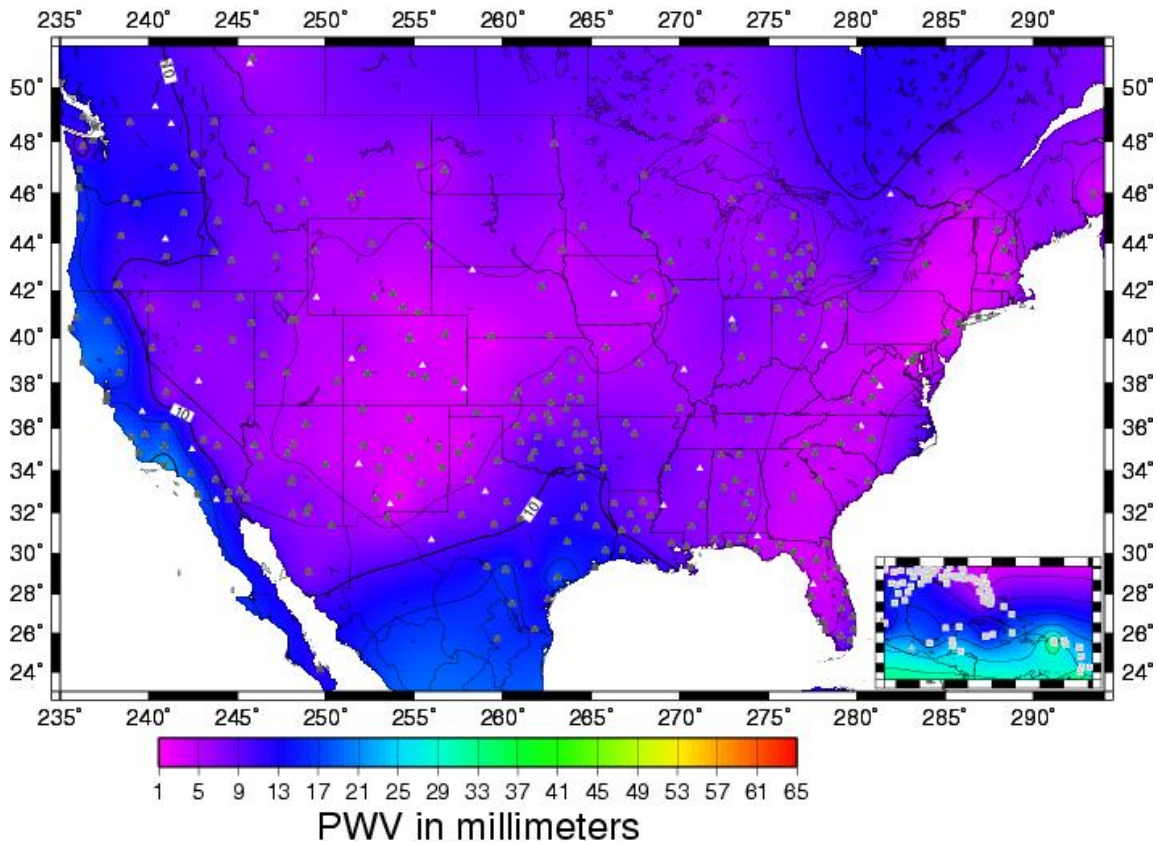
$$\mathrm{mf}(e) = \frac{1 + \cfrac{a}{1 + \cfrac{b}{1+c}}}{\sin e + \cfrac{a}{\sin e + \cfrac{b}{\sin e + c}}}$$

  – Approx. for layered atmosphere
- Path delay at some elevation angle e is ZTD*mf(e)
- VMF mapping functions vary with space and time based on distribution of water vapor.
- GMF is seasonal average of VMF



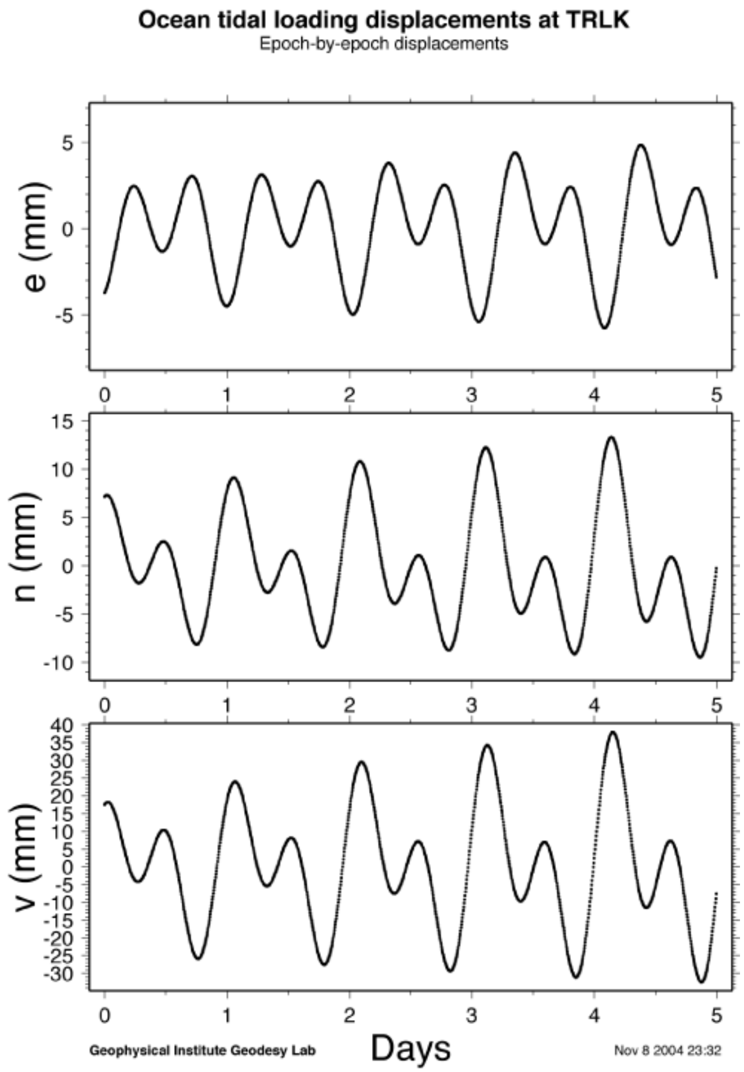
Map of Precipitable Water Vapor

PWV 22h-23h 02/05/09



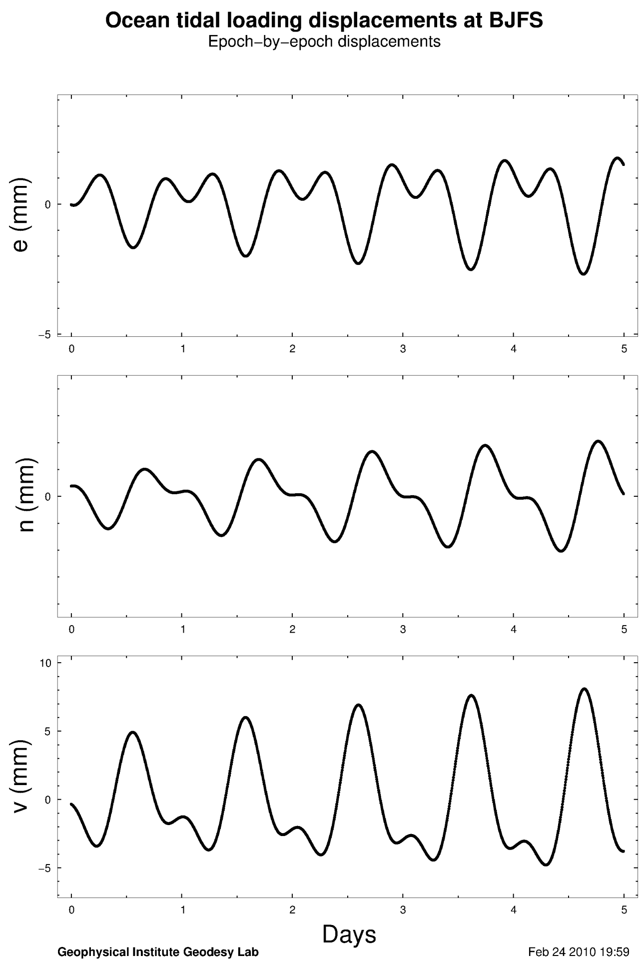
PWV is proportional to the path delay, given pressure, temperature

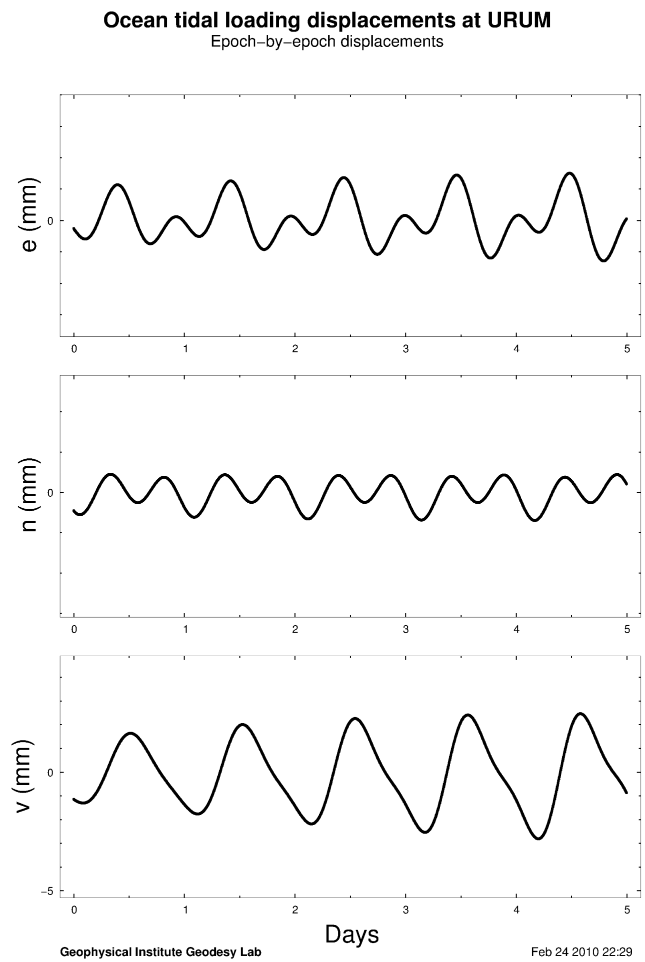
Ocean Tidal Loading



- Solid earth responds to changing load of ocean tides
- Displacements large near coast, where tidal range is large
- Details depend on ocean tides, coastline
- Accurate removal depends on good tidal models

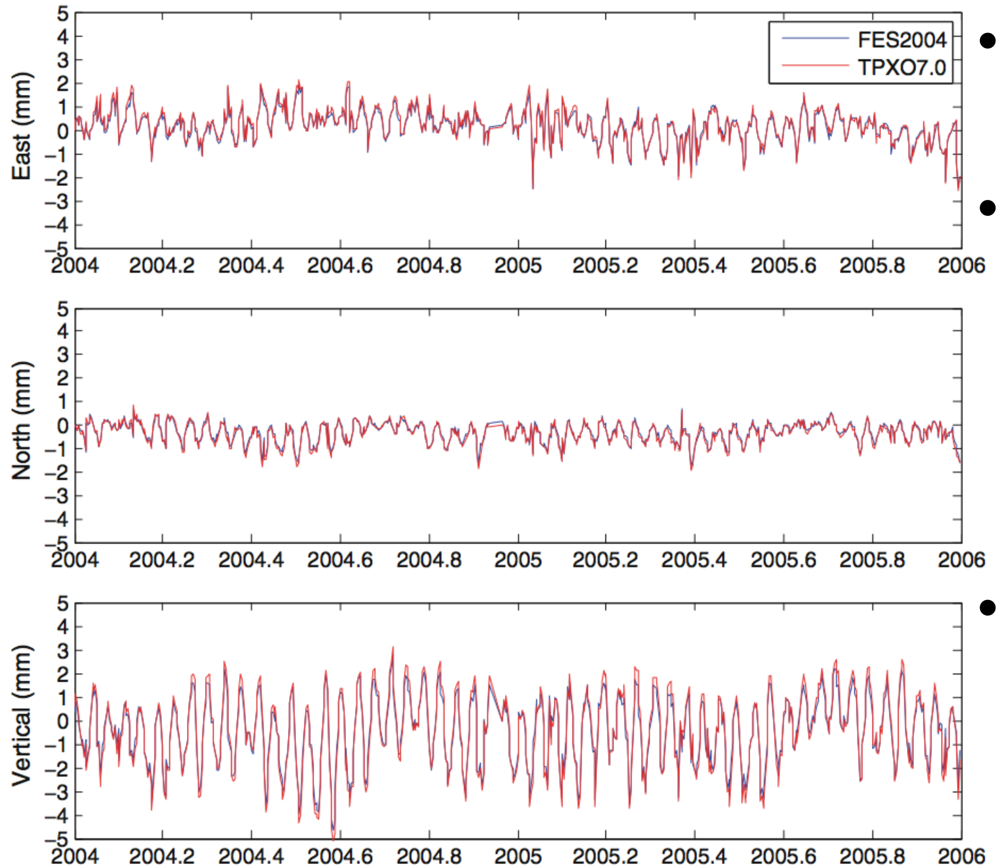
Compare to China





Computed in Center of Mass of Solid Earth frame

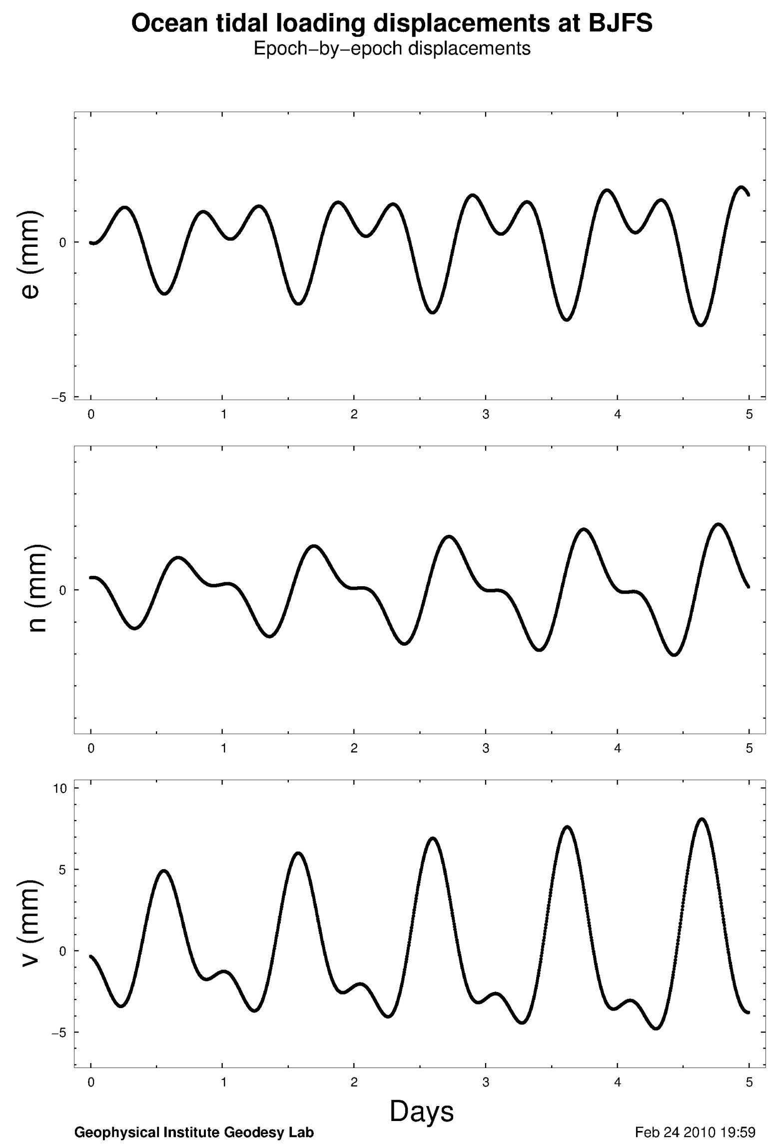
Frame Used for OTL Models is Critical



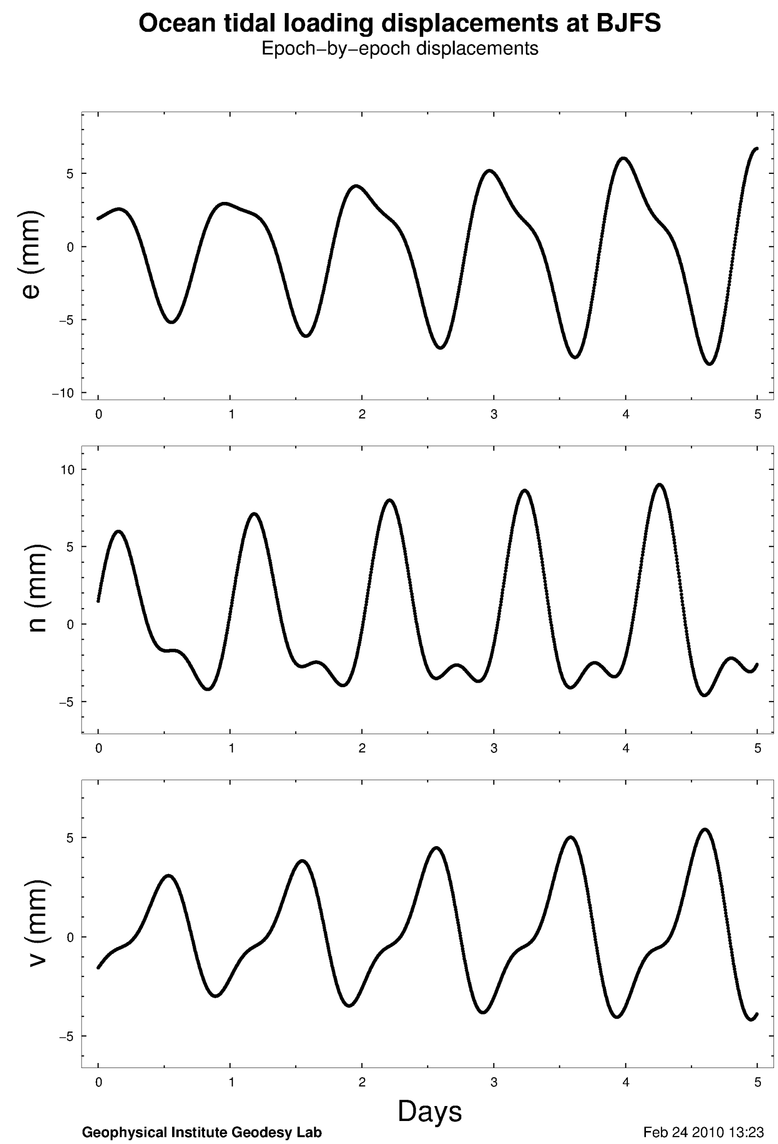
- Modern tidal models are very similar
- Can compute displacements in center of mass of solid earth (CE) or center of mass of earth + fluids (CM)
- Frame used for OTL computation makes a big difference.

Comparison of Frames

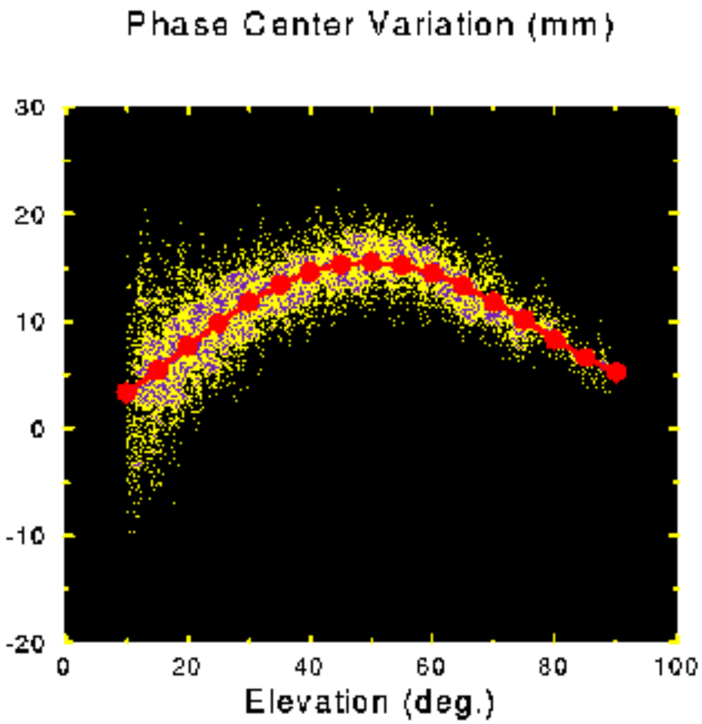
CE frame



CM frame



Antenna Phase Center Models



- Ideally, phase center is a point in space.
- Different for every type of antenna
- In reality, the phase center depends on the azimuth and elevation of incoming signal.
- Models assume azimuthal symmetry and fit elevation-dependence

Ambiguity Resolution

- Ambiguity resolution is a trick that can dramatically improve position quality for short surveys or kinematic positioning.
- If you know the ambiguity is an integer, and can determine which integer, then you can fix the ambiguity to that integer value.
  - Removing the ambiguity parameter dramatically improves the strength of the data to be used for determining the position
- We' ll talk about this more in the kinematic discussion